Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $\ell$, $\mathfrak{g}$ the affine Lie algebra associated with $\mathfrak{g}$:

$$
\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D,
$$

where $K$ is the central element and $D$ is the degree operator.

A weight $\lambda$ of $\mathfrak{g}$ is called a critical weight if $\langle \lambda + \rho, K \rangle = 0$. A critical weight $\lambda$ is called generic if

$$
\langle \lambda + \rho, \alpha^\vee \rangle \not\in \mathbb{Z}_{>0} \text{ for all } \alpha \in \Delta^r_+,
$$

where $\Delta^r_+$ is the set of positive real roots of $\mathfrak{g}$.

Let $L(\lambda)$ be the irreducible $\mathfrak{g}$-module of highest weight $\lambda$.

**Theorem 1.** If $\lambda$ is a generic critical weight, then

$$
\text{ch} L(\lambda) = e^\lambda \prod_{\alpha \in \Delta^r_+} (1 - e^{-\alpha})^{-1}.
$$

Theorem 1 was conjectured by V. Kac and D. Kazhdan [KK]. It has been proved by different methods: for $\mathfrak{sl}_2$ by M. Wakimoto [Wk], N. Wallach [Wl]; for the classical type affine Lie algebras by T. Hayashi [H] and R. Goodman and N. Wallach [GW]; for general affine Lie algebras by B. Feigin and E. Frenkel [FF1], J. M. Ku [Ku] and G. Kuroki [Kr]. See also the results in finite characteristic by O. Mathieu [M]; for affine Lie superalgebras by M. Gorelik [G].

The purpose of this note is to give a yet another proof of Theorem 1. We show that Theorem 1 easily follows as a corollary of a general result [A2] on the quantized Drinfeld-Sokolov reduction [FF2, FKW].

1.1. **Preliminaries.** Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathfrak{n}_+$ be a triangular decomposition of $\mathfrak{g}$. $\Delta$ the set of roots of $\mathfrak{g}$, $\Delta_+$ the set of positive roots of $\mathfrak{g}$. We also fix roots vectors $x_\alpha \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta$.

The Lie algebra $\mathfrak{g}$ is naturally identified with $\mathfrak{g} \otimes \mathbb{C} \subset \mathfrak{g}$. Then

$$
\mathfrak{h} = \mathfrak{h}^* \oplus \mathbb{C}K \oplus \mathbb{C}D
$$

is the standard Cartan subalgebra of $\mathfrak{g}$. We have $\mathfrak{h}^* = \mathfrak{h}^* \oplus \mathfrak{c}\Lambda_0 \oplus \mathfrak{c}\delta$, where $\Lambda_0$ and $\delta$ are dual elements of $K$ and $D$, respectively. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in \mathfrak{h}^*$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Let $\Delta_+$ be the set of positive roots of $\mathfrak{g}$. Then $\Delta_+ = \Delta^r_+ \sqcup \Delta^i_+$, where $\Delta^i_+ = \{ n\delta; n \in \mathbb{Z}_{>0} \}$ is the set of positive imaginary roots. The normalized invariant inner product ( , ) of $\mathfrak{g}$ naturally extends to $\mathfrak{g}$.

Let $M(\lambda)$ be the Verma module of highest weight $\lambda$. Then $L(\lambda)$ is the unique irreducible quotient of $M(\lambda)$.

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2. The details of the proof are given in [Fr, Theorem 4.8]
Theorem 2 ([KK]). Let $\lambda, \mu \in \mathfrak{h}^*$. Then the following are equivalent:

(i) $L(\mu)$ appears as a subquotient of $M(\lambda)$;
(ii) There exists a sequence of positive roots $\{\beta_k\}_{k=1}^r$, a sequence of positive integers $\{n_k\}_{k=1}^r$ and a sequence of weights $\{\lambda_k\}_{k=1}^r$ such that $\lambda_0 = \lambda$, $\lambda_r = \mu$ and $\lambda_k = \lambda_{k-1} - n_k \beta_k$ with $2(\beta_k, \lambda_{k-1} + \rho) = n_k(\beta_k, \beta_k)$ for $k = 1, \ldots, r$.

The following is easy to see.

Lemma 3. A critical weight $\lambda$ is generic if and only if $\lambda + n\delta$ is a generic critical weight for every $n \in \mathbb{Z}$.

By Theorem 2 and Lemma 3, we have the following assertion.

Proposition 4. Let $\lambda$ be a generic critical weight and $\mu \in \mathfrak{h}^*$. Then the following are equivalent:

(i) $L(\mu)$ appears as a subquotient of $M(\lambda)$;
(ii) $\mu = \lambda - n\delta$ for some $n \in \mathbb{Z}_{\geq 0}$.

Let $O_{-h^\vee}$ be the category $O$ ([K]) of $\mathfrak{g}$ at the critical level $-h^\vee$. For an object $V$ of $O_{-h^\vee}$, let $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$ be the weight space decomposition. The formal character $\text{ch} V$ of $V$ is defined as $\text{ch} V = \sum (\dim_c V^\lambda) e^\lambda$.

Let $\lambda$ be a generic critical weight. Then by Lemma 3 and Proposition 4 we have

\begin{equation}
\text{ch} L(\lambda) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda,n} \text{ch} M(\lambda - n\delta)
\end{equation}

with some $c_{\lambda,n} \in \mathbb{Z}$. Since

\[\text{ch} M(\mu) = e^\mu \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} \prod_{i \geq 1} (1 - q^{-i})^{-\ell},\]

where $q = e^{\delta}$, one has

\begin{equation}
\text{ch} L(\lambda) = e^\lambda \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda,n} q^{-n} \prod_{i \geq 1} (1 - q^{-i})^{-\ell}.
\end{equation}

1.2. A result from [A2]. Let

\[H^i(V) = H^{\mathfrak{n}+i}(L\mathfrak{n}_-, V \otimes \mathbb{C}\chi) \text{ with } i \in \mathbb{Z}\]

for an object $V$ of $O_{-h^\vee}$. Here, $L\mathfrak{n}_- = \mathfrak{n}_- \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{g}$ and $\chi$ is the character of $L\mathfrak{n}_-$ defined by

\[\chi(x_{-\alpha} \otimes t^n) = \begin{cases} 1 & \text{if } n = 0 \text{ and } \alpha \text{ is simple} \\ 0 & \text{otherwise}. \end{cases}\]

Also, $H^{\mathfrak{n}+i}(L\mathfrak{n}_-, V)$ is the semi-infinite $L\mathfrak{n}_-$-cohomology with coefficient in the $L\mathfrak{n}_-$-module $V$ ([Fr]). The degree operator $D$ naturally acts on the space $H^i(V)$ and one has

\[H^i(V) = \bigoplus_{d \in \mathbb{C}} H^i(V)_d,\]

where $H^i(V)_d = \{ v \in H^i(V); Dv = dv \}$ (see [A2] for details). Set

\[\text{ch} H^i(V) = \sum_{d \in \mathbb{C}} (\dim_c H^i(V)_d) q^d.\]
whenever it is well-defined.

**Proposition 5.** We have \( \text{ch} H^0(M(\lambda)) = q^{\lambda(D)} \prod_{i \geq 1} (1 - q^{-i})^{-\ell} \) for all \( \lambda \).

Proposition 5 is essentially proved in [FB] based on the idea of [dBT] (see [A1, Remark 5.8]).

By [A2, Main Theorem 1] we have the following assertion.

**Theorem 6.**

(i) We have \( H^i(V) = \{0\} \) with \( i \neq 0 \) for all objects \( V \) of \( O_{-\lambda'} \).

(ii) Let \( \lambda \) be a critical weight. If the classical part of \( \lambda \) is anti-dominant, then

\[
\text{ch} H^0(L(\lambda)) = q^{\lambda(D)}.
\]

Otherwise \( H^0(L(\lambda)) = \{0\} \).

Here, in Theorem 6, the classical part \( \lambda \) of \( \lambda \in \mathfrak{h}^* \) is the image of the projection \( \mathfrak{h}^* \to \mathfrak{h}^* \), and it is called anti-dominant if \( (\lambda + \rho, \alpha^\vee) \notin \mathbb{Z}_{>0} \) for all \( \alpha \in \Delta_+ \). In particular, the classical part of a generic critical weight is anti-dominant.

### 1.3. Proof of Theorem 1

From (1) and Theorem 6 (i), it follows that

\[
\text{ch} H^0(L(\lambda)) = \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda,n} \text{ch} H^0(M(\lambda - n\delta))
\]

Thus by Proposition 5 and Theorem 6 (ii), (iii) one obtains

\[
q^{\lambda(D)} = q^{\lambda(D)} \sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda,n} q^{-n} \prod_{i \geq 1} (1 - q^{-i})^{-\ell}.
\]

Thus

\[
\sum_{n \in \mathbb{Z}_{\geq 0}} c_{\lambda,n} q^{-n} \prod_{j \geq 1} (1 - q^{-i})^{-\ell} = 1.
\]

But then the theorem follows from (2).

\[\square\]

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### References


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