Trace identities of twisted Hecke operators
on the spaces of cusp forms of half-integral weight

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Abstract: Let $R_\psi$ be a twisting operator for a quadratic primitive character $\psi$ and $\tilde{T}(n^2)$ the $n^2$-th Hecke operator of half-integral weight. When $\psi$ has an odd conductor, we already found trace identities between twisted Hecke operators $R_\psi \tilde{T}(n^2)$ of half-integral weight and certain Hecke operators of integral weight for almost all cases (cf. [U1–3]). In this paper, the restriction is removed and we give similar trace identities for every quadratic primitive character $\psi$, including the case that $\psi$ has an even conductor.

Key words: Trace identity; twisting operator; half-integral weight; Hecke operator; cusp form.

1. Introduction. Let $k$, $A$, and $N$ be positive integers with $4 \mid N$. We denote the space of cusp forms of weight $2k$, level $A$ and the trivial character by $S(2k, A)$. Let $\chi$ be an even quadratic character defined modulo $N$. We denote the space of cusp forms of weight $k+1/2$, level $N$, and character $\chi$ by $S(k+1/2, N, \chi)$.

In [Sh], Shimura had found “Shimura Correspondence”. That is an important correspondence from Hecke eigenforms in $S(k+1/2, N, \chi)$ to those in $S(2k, N/2)$.

From the existence of Shimura Correspondence, we can expect that there exist certain identities between traces of Hecke operators of weight $k+1/2$ and those of weight $2k$.

After pioneering works of Niwa [N] and Kohnen [K], we had generalized their results and had found such identities between traces of Hecke operators for almost all levels $N$ (cf. [U1], [U3]). Furthermore, we generalized these results for the twisted Hecke operators ([U2]).

We explain more precisely. Let $\psi$, $R_\psi$, and $\tilde{T}(n^2)$ be the same as the abstract. In the papers [U1], [U2], and [U4], we calculated the traces of twisted Hecke operators $R_\psi \tilde{T}(n^2)$ both on $S(k+1/2, N, \chi)$ and on Kohnen’s plus space $S(k+1/2, N, \chi)K$. Moreover, when the conductor of $\psi$ is odd, we found that the above traces are linear combinations of the traces of certain Hecke operators on the spaces $S(2k, N')$ ($N'$ runs over positive divisors of $N/2$) for almost all cases. However we missed the cases such that $\text{ord}_p(N)$ (the $2$-adic additive valuation of $N$) is equal to 6 and the conductor of $\chi$ is divisible by 8.

The purpose of this paper is to remove the above restriction. Namely, we report trace identities for all quadratic primitive characters $\psi$, including both the above missing cases of odd conductors and the cases of even conductors. Details will appear in [U5].

2. Notation. The notation in this paper is the same as in the previous paper [U1]. Hence see [U1] and [U2] for the details of notation. Here, we explain several notations for convenience.

Let $k$, $N$, $\chi$ be the same as above. For a prime number $p$, let $\text{ord}_p(\cdot)$ be the $p$-adic additive valuation with $\text{ord}_p(p) = 1$ and $\cdot \mid p$ the $p$-adic absolute value which is normalized with $|p|_p = 1$. For a real number $x$, $[x]$ means the greatest integer less than or equal to $x$. Let $a$ be a non-zero integer and $b$ a positive integer. We write $a \mid b^\infty$ if every prime factor of $a$ divides $b$.

Let $\rho$ be any Dirichlet character. We denote the conductor of $\rho$ by $f(\rho)$ and for any prime number $p$, the $p$-primary component of $\rho$ by $\rho_p$. Furthermore we set $\rho_A := \prod_{p \mid A} \rho_p$ for an arbitrary integer $A$. Here $p$ runs over all prime divisors of $A$. We denote by $(\cdot)$ the Kronecker symbol. See [M, p. 82] for a definition of this symbol.
Let $V$ be a finite-dimensional vector space over $C$. We denote the trace of a linear operator $T$ on $V$ by $\text{tr}(T; V)$.

Put $\mu := \text{ord}_2(N)$ and $\nu := \text{ord}_p(N)$ for any odd prime number $p$. Then we decompose $N = 2^\mu M$. Namely, $M$ is the odd part of $N$.

3. Results. Let $\psi$ be a quadratic primitive character with conductor $r$. Then we can express the conductor $r$ as follows:

$$r = 2^u L, \quad u = 0, 2, \text{ and } 3$$

and $L$ is a squarefree positive odd integer.

We consider the following conditions (*1)–(*3).

(*1) $L^2 \mid M$.

(*2) $L^2 \mid M$ and \[ \begin{cases} \mu \geq 5, \text{ if } f(\chi_2) = 8, \\ \mu \geq 4, \text{ if } f(\chi_2) \mid 4, \end{cases} \]

(*3) $L^2 \mid M$ and $\mu \geq 6$.

From now on until the end of this paper, we assume the following.

**Assumption.** We impose the condition (*1), (ii), or (iii) according to $u = 0, 2, \text{ or } 3$ respectively.

Now, let $R_\psi$ be the twisting operator of $\psi$: $f = \sum_{n \geq 1} a(n)q^n \mapsto f | R_\psi := \sum_{n \geq 1} a(n)\psi(n)q^n$.

$(q := \exp(2\pi \sqrt{-1}z), z \in C, \text{Im } z > 0)$.

Then, from the above conditions (1–3) and the assumption $\psi^2 = 1$, we see that the twisting operator $R_\psi$ fixes the space of cusp forms $S(k + 1/2, N; \chi)$ (cf. [Sh, Lemma 3.6]).

In the case of $k = 1$, we need to make a certain modification. It is well-known that the space $S(3/2, N; \chi)$ contains a subspace $U(N; \chi)$ which corresponds to a space of Eisenstein series via Shimura correspondence and which is generated by theta series of special type (cf. [U2, §00(c)]). Let $V(N; \chi)$ be the orthogonal complement of $U(N; \chi)$ in $S(3/2, N; \chi)$. Then it is also well-known that $V(N; \chi)$ corresponds to a space of cusp forms of weight 2 via Shimura correspondence. Hence we need to consider the subspace $V(N; \chi)$ in place of $S(3/2, N; \chi)$ in the case of $k = 1$. The subspaces $U(N; \chi)$ and $V(N; \chi)$ are fixed by the twisting operator $R_\psi$ (See [U5] for a proof and refer also to [U2, p.94]).

Moreover, the $n^2$-th Hecke operators $T(n^2)$, $(n, N) = 1$, also fix the subspace $V(N; \chi)$ (cf. [U1, p.508]).

Thus for any positive integer $n$ with $(n, N) = 1$, we can consider the twisted Hecke operator $R_\psi T(n^2)$ on the spaces $S(k + 1/2, N; \chi) (k \geq 2)$ and $V(N; \chi)$ (cf. [U2, p.86]).

For the statement of Theorem, we prepare a little more notation.

First we decompose the level $N$ with respect to $L$ as follows:

$$N = 2^\mu L_0 L_2, \quad L_0 > 0, \quad L_2 > 0,$$

$$\mu := \text{ord}_2(N), \quad L_0 \mid L^{\infty}, \quad (L_2, L) = 1.$$

And we put

$$N_0 := \prod_{p \mid L} p^{2(\nu_p - 1)/2} + 1.$$

Here $p$ runs over all prime divisors of $L$.

Next, let $A$ be an any positive integer. For any odd prime number $p$ and any integers $a, b$ $(0 \leq a \leq \text{ord}_p(A)/2)$, we put

$$\lambda_p(\chi_p, \text{ord}_p(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{b}{p}\right), & \text{if } 1 \leq a \leq [(\text{ord}_p(A) - 1)/2], \\ \chi_p(-b), & \text{if } \text{ord}_p(A) \text{ is even} \\ & \text{and } a = \text{ord}_p(A)/2 \geq 1. \end{cases}$$

And for any integers $a, b$ $(0 \leq a \leq \text{ord}_2(A)/2)$, we put

$$\lambda_2(\chi_2, \text{ord}_2(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{if } a = 1, \\ \xi(b)(1 + \left(\frac{b}{2}\right)), & \text{if } 2 \leq a \leq [(\text{ord}_2(A) - 1)/2], \\ \xi(b)\chi_2(-b), & \text{if } \text{ord}_2(A) \text{ is even} \\ & \text{and } a = \text{ord}_2(A)/2 \geq 2. \end{cases}$$

Here, $\xi(b) := (1 - \left(\frac{b}{2}\right))/2$.

Then for any integer $b$ and any square integer $c$, we put

$$\Lambda_\chi(\psi, A; b, c) := \prod_{(p, r) = 1} \lambda_p(\chi_p, \text{ord}_p(A); b, \text{ord}_p(c)/2).$$

Here $p$ runs over all prime divisors of $A$ prime to $r$.

Furthermore, let $B$ be a positive integer such that $B \mid r^{\infty}$ and $(A/B, B) = 1$. For all positive integers $n$ such that $(n, N) = 1$, we define
\[ \Theta_\psi[2k, n; A, B, \chi] = \Theta_\psi[A, B, \chi] = \sum_{0 < N_1 | A, N_1 = 1} \Lambda_k(\psi, r_n; N_1) \times \text{tr}(W(BN_1)T(n); S(2k, N_1 N_2)), \]

where \(N_1\) runs over all square divisors of \(A\) which are prime to \(r\) and \(N_2 := A \prod_{p | N_1} |A|_p\).

**Remark.** All the spaces which occur in the definition of \(\Theta_\psi[A, B, \chi]\) are contained in the space \(S(2k, A)\).

Finally, let \(\chi_r\) be the \(r\)-primary component of \(\chi\) and \(\chi'_r := \prod_{p | N_1(p,r) = 1} \chi_p\), where \(p\) runs over all prime divisors of \(N\) which are prime to \(r\). Then we put

\[ c(k, n; \psi, \chi) := \psi(-1)^k N_1 \chi_n(\chi'_r - r). \]

Under these notations, we can state trace identities of the twisted Hecke operators \(R_\psi T(n^2)\).

First we state trace identities for the case of odd conductors.

**Theorem 1.** Let \(k, N, \) and \(\chi\) be the same as above. Suppose that \(\psi\) is a quadratic primitive character defined modulo an odd positive integer \(r\). Hence we assume the condition \((\ast.1)\).

For all positive integers \(n\) such that \((n, N) = 1\), we have the following trace identities.

1. Suppose that \(\mu = 2\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
&= c(\psi, \chi) \Theta_\psi[2N_0 L_2, N_0, \chi].
\end{align*}
\]

2. Suppose that \(2 \leq \mu \leq 4\) and furthermore \(f(\chi_2) = 8\) if \(\mu = 4\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
&= 2c(\psi, \chi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi].
\end{align*}
\]

3. Suppose that \(4 \leq \mu \leq 6\) and furthermore \(f(\chi_2)\) divides 4 if \(\mu = 4, 6\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
&= 2c(\psi, \chi) \Theta_\psi[2^{\mu-2} N_0 L_2, N_0, \chi].
\end{align*}
\]

4. Suppose that \(\mu = 6\) and \(f(\chi_2) = 8\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & \text{if } k = 1
\end{cases}
\begin{cases}
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k \geq 2 \\
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k = 1
\end{cases}
\end{align*}
\]

5. Suppose that \(\mu = 7\) and \(f(\chi_2) = 16\) if \(\mu = 8\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^7 M, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^7 M; \chi)) & \text{if } k = 1
\end{cases}
\begin{cases}
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k \geq 2 \\
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k = 1
\end{cases}
\end{align*}
\]

6. Suppose that \(\mu = 7\) and \(f(\chi_2) = 8\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^7 M, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^7 M; \chi)) & \text{if } k = 1
\end{cases}
\begin{cases}
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k \geq 2 \\
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k = 1
\end{cases}
\end{align*}
\]

7. Suppose that \(\mu \geq 8\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^8 M, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^8 M; \chi)) & \text{if } k = 1
\end{cases}
\begin{cases}
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k \geq 2 \\
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k = 1
\end{cases}
\end{align*}
\]

8. Suppose that \(\mu = 8\) and \(f(\chi_2) = 16\). We have

\[
\begin{align*}
&\begin{cases}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^8 M, \chi)) & \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^8 M; \chi)) & \text{if } k = 1
\end{cases}
\begin{cases}
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k \geq 2 \\
\theta(\psi) \Theta_\psi[2^{\mu-1} N_0 L_2, N_0, \chi] & \text{if } k = 1
\end{cases}
\end{align*}
\]
Next, we state trace identities for the case of even conductor.

**Theorem 2.** Let $k$, $N$, and $\chi$ be the same as above. Suppose that $\psi$ is a quadratic primitive character defined modulo an even positive integer $v$. Hence we assume the condition $(\ast 2)$ or $(\ast 3)$ according to $v = 2$ or 3 respectively.

For all positive integers $n$ such that $(n, N) = 1$, we have the following trace identities.

**Case I.** $(v = 2)$ \( \Leftrightarrow \psi_2 = (\frac{-1}{n}) \)

(I-1) Suppose that $\mu = 4$ and $f(\chi_2) = 4$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^4 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^4 M, \chi)) & \quad \text{if } k = 1 \\
= \chi_2 \left( \frac{-1}{Ln} \right) c(\psi, \chi)\Theta_{\psi}\{2^5 N_0 L_2, 2^5 N_0, \chi].
\end{align*}
\]

(I-2) Suppose that $\mu = 5$ and $f(\chi_2) = 4$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^5 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^5 M, \chi)) & \quad \text{if } k = 1 \\
= \chi_2 \left( \frac{-1}{Ln} \right) c(\psi, \chi) \times \{\Theta_{\psi}\{2^5 N_0 L_2, 2^5 N_0, \chi] \\
- 2\Theta_{\psi}\{2^6 N_0 L_2, 2^6 N_0, \chi] + 2\Theta_{\psi}\{2^5 N_0 L_2, 2^5 N_0, \chi].
\end{align*}
\]

(I-3) Suppose that $\mu = 5, 6$ and $f(\chi_2) = 6$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^\mu M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^\mu M, \chi)) & \quad \text{if } k = 1 \\
= (1 - \psi(1)) \left( \frac{-1}{n} \right) c(\psi, \chi) \times \{\Theta_{\psi}\{2^\mu N_0 L_2, 2^\mu N_0, \chi] - \Theta_{\psi}\{2^\mu N_0 L_2, 2^\mu N_0, \chi].
\end{align*}
\]

(I-4) Suppose that $\mu = 7$ and $f(\chi_2) = 8$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^\mu M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^\mu M, \chi)) & \quad \text{if } k = 1 \\
= (1 - \psi(1)) \left( \frac{-1}{n} \right) c(\psi, \chi) \times \Theta_{\psi}\{2^{\mu-2} N_0 L_2, 2^{\mu-2} N_0, \chi].
\end{align*}
\]

(I-5) Suppose that $\mu \geq 8$, or $\mu = 6, 7$ and $f(\chi_2) = 4$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^\mu M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^\mu M, \chi)) & \quad \text{if } k = 1 \\
= (1 - \psi(1)) \left( \frac{-1}{n} \right) c(\psi, \chi) \times \Theta_{\psi}\{2^{\mu-2} N_0 L_2, 2^{\mu-2} N_0, \chi].
\end{align*}
\]

Here $\bar{\mu}$ is the greatest even integer less than or equal to $\mu$, i.e. $\bar{\mu} = 2[\mu/2]$.

**Case II.** $(v = 3)$ \( \Leftrightarrow \psi_3 = (\frac{-1}{n}) \)

(II-1) Suppose that $\mu = 6, 7$ and $f(\chi_2) = 8$.

We have
\[
\begin{align*}
\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^\mu M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^\mu M, \chi)) & \quad \text{if } k = 1 \\
= (1 - \psi(1)) \left( \frac{-1}{n} \right) c(\psi, \chi) \times \Theta_{\psi}\{2^{\mu-2} N_0 L_2, 2^{\mu-2} N_0, \chi].
\end{align*}
\]

Here $\bar{\mu}$ is the greatest odd integer less than or equal to $\mu$, i.e. $\bar{\mu} = 2[(\mu - 1)/2] + 1$.

**4. Concluding remarks.** We can expect to establish a theory of newforms by using these trace identities. In fact, we established a theory of newforms in the case of level $2^n$. See [U6] for the results.

References


