Trace identities of twisted Hecke operators on the spaces of cusp forms of half-integral weight

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Abstract: Let $R_\psi$ be a twisting operator for a quadratic primitive character $\psi$ and $\tilde{T}(n^2)$ the $n^2$-th Hecke operator of half-integral weight. When $\psi$ has an odd conductor, we already found trace identities between twisted Hecke operators $R_\psi \tilde{T}(n^2)$ of half-integral weight and certain Hecke operators of integral weight for almost all cases (cf. [U1–3]). In this paper, the restriction is removed and we give similar trace identities for every quadratic primitive character $\psi$, including the case that $\psi$ has an even conductor.

Key words: Trace identity; twisting operator; half-integral weight; Hecke operator; cusp form.

1. Introduction. Let $k$, $A$, and $N$ be positive integers with $4 \mid N$. We denote the space of cusp forms of weight $2k$, level $A$ and the trivial character by $S(2k, A)$. Let $\chi$ be an even quadratic character defined modulo $N$. We denote the space of cusp forms of weight $k + 1/2$, level $N$, and character $\chi$ by $S(k + 1/2, N, \chi)$.

In [Sh], Shimura had found “Shimura Correspondence”. That is an important correspondence from Hecke eigenforms in $S(k + 1/2, N, \chi)$ to those in $S(2k, N/2)$.

From the existence of Shimura Correspondence, we can expect that there exist certain identities between traces of Hecke operators of weight $k + 1/2$ and those of weight $2k$.

After pioneering works of Niwa [N] and Kohnen [K], we had generalized their results and had found such identities between traces of Hecke operators for almost all levels $N$ (cf. [U1], [U3]). Furthermore, we generalized these results for the twisted Hecke operators ([U2]).

We explain more precisely. Let $\psi$, $R_\psi$, and $\tilde{T}(n^2)$ be the same as in the abstract. In the papers [U1], [U2], and [U4], we calculated the traces of twisted Hecke operators $R_\psi \tilde{T}(n^2)$ both on $S(k + 1/2, N, \chi)$ and on Kohnen’s plus space $S(k + 1/2, N, \chi)_K$. Moreover, when the conductor of $\psi$ is odd, we found that the above traces are linear combinations of the traces of certain Hecke operators on the spaces $S(2k, N')$ ($N'$ runs over positive divisors of $N/2$) for almost all cases. However we missed the cases such that $\text{ord}_2(N)$ (the 2-adic additive valuation of $N$) is equal to 6 and the conductor of $\chi$ is divisible by 8.

The purpose of this paper is to remove the above restriction. Namely, we report trace identities for all quadratic primitive characters $\psi$, including both the above missing cases of odd conductors and the cases of even conductors. Details will appear in [U5].

2. Notation. The notation in this paper is the same as in the previous paper [U1]. Hence see [U1] and [U2] for the details of notation. Here, we explain several notations for convenience.

Let $k$, $N$, $\chi$ be the same as above. For a prime number $p$, let $\text{ord}_p(\cdot)$ be the $p$-adic additive valuation with $\text{ord}_p(p) = 1$ and $|\cdot|_p$ the $p$-adic absolute value which is normalized with $|p|_p = 1$. For a real number $x$, $\lfloor x \rfloor$ means the greatest integer less than or equal to $x$. Let $a$ be a non-zero integer and $b$ a positive integer. We write $a \mid b^\infty$ if every prime factor of $a$ divides $b$.

Let $\rho$ be any Dirichlet character. We denote the conductor of $\rho$ by $f(\rho)$ and for any prime number $p$, the $p$-primary component of $\rho$ by $\rho_p$. Furthermore we set $\rho_A := \prod_{p \mid A} \rho_p$ for an arbitrary integer $A$. Here $p$ runs over all prime divisors of $A$. We denote by $(\cdot)$ the Kronecker symbol. See [M, p. 82] for a definition of this symbol.
Let $V$ be a finite-dimensional vector space over $C$. We denote the trace of a linear operator $T$ on $V$ by $\text{tr}(T; V)$.

Put $\mu := \text{ord}_2(N)$ and $\nu := \text{ord}_p(N)$ for any odd prime number $p$. Then we decompose $N = 2^\mu M$. Namely, $M$ is the odd part of $N$.

3. Results. Let $\psi$ be a quadratic primitive character with conductor $r$. Then we can express the conductor $r$ as follows:

$$\begin{cases}
r = 2^u L, & u = 0, 2, \text{ and } 3 \\
\text{and } L \text{ is a squarefree positive odd integer.}
\end{cases}$$

We consider the following conditions (1)–(3).

(1) $L^2 | M$.

(2) $L^2 | M$ and \( \mu \geq 5 \), if $f(\chi_2) = 8$.

(3) $L^2 | M$ and $\mu \geq 6$.

From now on until the end of this paper, we assume the following.

Assumption. We impose the condition (1), (2), or (3) according to $u = 0, 2, \text{ or } 3$ respectively.

Now, let $R_\psi$ be the twisting operator of $\psi$:

$$f = \sum_{n \geq 1} a(n)q^n \mapsto f | R_\psi := \sum_{n \geq 1} a(n)\psi(n)q^n,$$

$$(q := \exp(2\pi i z)), \ z \in C, \ \text{Im } z > 0).$$

Then, from the above conditions (1–3) and the assumption $\psi^2 = 1$, we see that the twisting operator $R_\psi$ fixes the space of cusp forms $S(k + 1/2, N, \chi)$ (cf. [Sh, Lemma 3.6]).

In the case of $k = 1$, we need to make a certain modification. It is well-known that the space $S(3/2, N, \chi)$ contains a subspace $U(N; \chi)$ which corresponds to a space of Eisenstein series via Shimura correspondence and which is generated by the theta series of special type (cf. [U2, §0(c)]) Let $V(N; \chi)$ be the orthogonal complement of $U(N; \chi)$ in $S(3/2, N, \chi)$. Then it is also well-known that $V(N; \chi)$ corresponds to a space of cusp forms of weight 2 via Shimura correspondence. Hence we need to consider the subspace $V(N; \chi)$ in place of $S(3/2, N, \chi)$ in the case of $k = 1$. The subspaces $U(N; \chi)$ and $V(N; \chi)$ are fixed by the twisting operator $R_\psi$ (See [U5] for a proof and refer also to [U2, p. 94]). Moreover, the $n^2$-th Hecke operators $\tilde{T}(n^2)$, $(n, N) = 1$, also fix the subspace $V(N; \chi)$ (cf. [U1, p. 508]).

Thus for any positive integer $n$ with $(n, N) = 1$, we can consider the twisted Hecke operator $R_\psi \tilde{T}(n^2)$ on the spaces $S(k + 1/2, N, \chi) \ (k \geq 2)$ and $V(N; \chi) \ (k = 1)$ (cf. [U2, p. 86]).

For the statement of Theorem, we prepare a little more notation.

First we decompose the level $N$ with respect to $L$ as follows:

$$N = 2^\mu L_0 L_2, \ \ L_0 > 0, \ \ L_2 > 0,$$

$$\mu := \text{ord}_2(N), \ \ L_0 | L^\infty, \ \ (L_2, L) = 1.$$ And we put

$$N_0 := \prod_{p | L} p^{2(\nu_p - 1)/2} + 1.$$ Here $p$ runs over all prime divisors of $L$.

Next, let $A$ be any positive integer. For any odd prime number $p$ and any integers $a, b \ (0 \leq a \leq \text{ord}_p(A)/2)$, we put

$$\lambda_p(\chi_p, \text{ord}_p(A); b, a) := \begin{cases}
1, & \text{if } a = 0, \\
1 + \left(\frac{b}{p}\right), & \text{if } 1 \leq a \leq [(\text{ord}_p(A) - 1)/2], \\
\chi_p(-b), & \text{if } \text{ord}_p(A) \text{ is even, and } a = \text{ord}_p(A)/2 \geq 1.
\end{cases}$$

And for any integers $a, b \ (0 \leq a \leq \text{ord}_2(A)/2)$, we put

$$\lambda_2(\chi_2, \text{ord}_2(A); b, a) := \begin{cases}
1, & \text{if } a = 0, \\
0, & \text{if } a = 1, \\
\xi(b)(1 + \left(\frac{b}{2}\right)), & \text{if } 2 \leq a \leq [(\text{ord}_2(A) - 1)/2], \\
\xi(b)\chi_2(-b), & \text{if } \text{ord}_2(A) \text{ is even, and } a = \text{ord}_2(A)/2 \geq 2.
\end{cases}$$

Here, $\xi(b) := (1 - (\frac{b}{2})) / 2$.

Then for any integer $b$ and any square integer $c$, we put

$$\Lambda_\chi(\psi, A; b, c) := \prod_{p | A} \lambda_p(\chi_p, \text{ord}_p(A); b, \text{ord}_p(c)/2).$$ Here $p$ runs over all prime divisors of $A$ prime to $r$.

Furthermore, let $B$ be a positive integer such that $B \mid r^\infty$ and $(A/B, B) = 1$. For all positive integers $n$ such that $(n, N) = 1$, we define
\[ \Theta_\psi[2k, n; A, B, \chi] = \Theta_\psi[A, B, \chi] \]
\[ := \sum_{0 \leq N_1 | A, N_1 = 1} \Lambda_\psi(\psi, n_1) \times \operatorname{tr}(W(BN_1)T(n); S(2k, N_1N_2)), \]
where \( N_1 \) runs over all square divisors of \( A \) which are prime to \( r \) and \( N_2 := A \prod_{p | N_1} |A|_p \).

**Remark.** All the spaces which occur in the definition of \( \Theta_\psi[A, B, \chi] \) are contained in the space
\( S(2k, A) \).

Finally, let \( \chi_r \) be the \( r \)-primary component of \( \chi \) and \( \chi_r' := \prod_{p | N, (p, r) = 1} \chi_p \), where \( p \) runs over all prime divisors of \( N \) which are prime to \( r \). Then we put
\[ c(k, n; \psi, \chi) = c(\psi, \chi) := \psi(-1)^k \chi_r(n_1) \chi_r'(-r). \]

Under these notations, we can state trace identities of the twisted Hecke operators \( R_\psi T(n^2) \).

First we state trace identities for the case of odd conductors.

**Theorem 1.** Let \( k, N, \) and \( \chi \) be the same as above. Suppose that \( \psi \) is a quadratic primitive character defined modulo an odd positive integer \( r \). Hence we assume the condition \((*1)\).

For all positive integers \( n \) such that \( (n, N) = 1 \), we have the following trace identities.

1. Suppose that \( \mu = 2 \). We have
\[
\begin{aligned}
& \begin{cases} 
\operatorname{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2, \\
\operatorname{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
= c(\psi, \chi) \Theta_\psi[N_0 L_2, N_0, \chi].
\end{aligned}
\]

2. Suppose that \( 2 \leq \mu \leq 4 \) and furthermore \( f(\chi_2) = 8 \) if \( \mu = 4 \). We have
\[
\begin{aligned}
& \begin{cases} 
\operatorname{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2, \\
\operatorname{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
= c(\psi, \chi) \Theta_\psi[2^{\mu-3} N_0 L_2, N_0, \chi].
\end{aligned}
\]

3. Suppose that \( 4 \leq \mu \leq 6 \) and furthermore \( f(\chi_2) \) divides 4 if \( \mu = 4, 6 \). We have
\[
\begin{aligned}
& \begin{cases} 
\operatorname{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, N, \chi)) & \text{if } k \geq 2, \\
\operatorname{tr}(R_\psi \tilde{T}(n^2); V(N; \chi)) & \text{if } k = 1
\end{cases} \\
= 2c(\psi, \chi) \Theta_\psi[2^{\mu-2} N_0 L_2, N_0, \chi].
\end{aligned}
\]

4. Suppose that \( \mu = 6 \) and \( f(\chi_2) = 8 \). We have
\[
\begin{aligned}
& \begin{cases} 
\operatorname{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^n M, \chi)) & \text{if } k \geq 2, \\
\operatorname{tr}(R_\psi \tilde{T}(n^2); V(2^n M; \chi)) & \text{if } k = 1
\end{cases} \\
= 2c(\psi, \chi) \Theta_\psi[2^{\mu-6} N_0 L_2, N_0, \chi].
\end{aligned}
\]
Next, we state trace identities for the case of even conductor.

**Theorem 2.** Let $k$, $N$, and $\chi$ be the same as above. Suppose that $\psi$ is a quadratic primitive character defined modulo an even positive integer $r$. Hence we assume the condition $(\ast 2)$ or $(\ast 3)$ according to $u = 2$ or $3$ respectively.

For all positive integers $n$ such that $(n, N) = 1$, we have the following trace identities.

(1) Suppose that $u = 2$. Let $\mu$ be the greatest even integer less than or equal to $\mu$, i.e. $\bar{\mu} = 2[\mu/2]$.

(1-1) Suppose that $\mu = 4$ and $f(\chi_2)$ divides 4.

We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^4M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^4M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= \chi_2 \left( \frac{1}{L_n} \right) c(\psi, \chi) \Theta_\psi[2^4N_0L_2, 2^2N_0, \chi] \\
- 2\Theta_\psi[2^2N_0L_2, N_0, \chi] + 2\Theta_\psi[2^2N_0L_2, 2^2N_0, \chi].
\end{align*}

(1-2) Suppose that $\mu = 5$ and $f(\chi_2)$ divides 4.

We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^5M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^5M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= \chi_2 \left( \frac{1}{L_n} \right) c(\psi, \chi) \times \Theta_\psi[2^5N_0L_2, N_0, \chi] \\
- 2\Theta_\psi[2^3N_0L_2, N_0, \chi] + 2\Theta_\psi[2^3N_0L_2, 2^2N_0, \chi].
\end{align*}

(1-3) Suppose that $\mu = 6, \text{and } f(\chi_2) = 8$. We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^6M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^6M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= (1 - \psi(-1) \left( \frac{1}{n} \right)) c(\psi, \chi) \\
\times \Theta_\psi[2^6N_0L_2, 2^6N_0, \chi] - \Theta_\psi[2^4N_0L_2, 2^4N_0, \chi].
\end{align*}

(1-4) Suppose that $\mu = 7$ and $f(\chi_2) = 8$. We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^7M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^7M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= (1 - \psi(-1) \left( \frac{1}{n} \right)) c(\psi, \chi) \\
\times \Theta_\psi[2^7N_0L_2, 2^7N_0, \chi] - \Theta_\psi[2^5N_0L_2, 2^4N_0, \chi].
\end{align*}

(1-5) Suppose that $\mu \geq 8$, or $\mu = 6, \text{and } f(\chi_2)$ divides 4.

We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^\mu M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^\mu M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= (1 - \psi(-1) \left( \frac{1}{n} \right)) c(\psi, \chi) \\
\times \Theta_\psi[2^\mu-2N_0L_2, 2^\mu-2N_0, \chi].
\end{align*}

Here $\bar{\mu}$ is the greatest even integer less than or equal to $\mu$, i.e. $\bar{\mu} = 2[\mu/2]$.

(II-1) Suppose that $\mu = 6, \text{and } f(\chi_2) = 8$. We have
\[
\begin{aligned}
\{\text{tr}(R_\psi T(n^2); S(k + 1/2, 2^6M, \chi)) &\quad \text{if } k \geq 2 \\
\text{tr}(R_\psi T(n^2); V(2^6M, \chi)) &\quad \text{if } k = 1
\end{aligned}
\]

\begin{align*}
= (1 - \psi(-1) \left( \frac{1}{n} \right)) c(\psi, \chi) \\
\times \Theta_\psi[2^6N_0L_2, 2^6N_0, \chi].
\end{align*}

Here $\bar{\mu}$ is the greatest odd integer less than or equal to $\mu$, i.e. $\bar{\mu} = 2[(\mu - 1)/2] + 1$.

4. **Concluding remarks.** We can expect to establish a theory of newforms by using these trace identities. In fact, we established a theory of newforms in the case of level $2^n$. See [U6] for the results.

**References**