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なお、この論文は弦型拡張Hecke演算子と一階フジタ多様体の点のトレイスを示す手法の確立を目的としています。
Trace identities of twisted Hecke operators on the spaces of cusp forms of half-integral weight

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Abstract: Let \( R_\psi \) be a twisting operator for a quadratic primitive character \( \psi \) and \( \tilde{T}(n^2) \) the \( n^2 \)-th Hecke operator of half-integral weight. When \( \psi \) has an odd conductor, we already found trace identities between twisted Hecke operators \( R_\psi \tilde{T}(n^2) \) of half-integral weight and certain Hecke operators of integral weight for almost all cases (cf. [U1–3]). In this paper, the restriction is removed and we give similar trace identities for every quadratic primitive character \( \psi \), including the case that \( \psi \) has an even conductor.

Key words: Trace identity; twisting operator; half-integral weight; Hecke operator; cusp form.

1. Introduction. Let \( k \), \( A \), and \( N \) be positive integers with \( 4 \mid N \). We denote the space of cusp forms of weight \( 2k \), level \( A \) and the trivial character by \( S(2k,A) \). Let \( \chi \) be an even quadratic character defined modulo \( N \). We denote the space of cusp forms of weight \( k+1/2 \), level \( N \), and character \( \chi \) by \( S(k+1/2,N,\chi) \).

In [Sh], Shimura had found “Shimura Correspondence”. That is an important correspondence between Hecke eigenforms in \( S(k+1/2,N,\chi) \) to those in \( S(2k,N/2) \).

From the existence of Shimura Correspondence, we can expect that there exist certain identities between traces of Hecke operators of weight \( k+1/2 \) and those of weight \( 2k \).

After pioneering works of Niwa [N] and Kohnen [K], we had generalized their results and had found such identities between traces of Hecke operators for almost all levels \( N \) (cf. [U1], [U3]). Furthermore, we generalized these results for the twisted Hecke operators ([U2]).

We explain more precisely. Let \( \psi \), \( R_\psi \), and \( \tilde{T}(n^2) \) be the same as the abstract. In the papers [U1], [U2], and [U4], we calculated the traces of twisted Hecke operators \( R_\psi \tilde{T}(n^2) \) both on \( S(k+1/2,N,\chi) \) and on Kohnen’s plus space \( S(k+1/2,N,\chi)_K \). Moreover, when the conductor of \( \psi \) is odd, we found that the above traces are linear combinations of the traces of certain Hecke operators on the spaces \( S(2k,N') \) (\( N' \) runs over positive divisors of \( N/2 \)) for almost all cases. However we missed the cases such that \( \text{ord}_2(N) \) (the 2-adic additive valuation of \( N \)) is equal to 6 and the conductor of \( \chi \) is divisible by 8.

The purpose of this paper is to remove the above restriction. Namely, we report trace identities for all quadratic primitive characters \( \psi \), including both the above missing cases of odd conductors and the cases of even conductors. Details will appear in [U5].

2. Notation. The notation in this paper is the same as in the previous paper [U1]. Hence see [U1] and [U2] for the details of notation. Here, we explain several notations for convenience.

Let \( k \), \( N \), \( \chi \) be the same as above. For a prime number \( p \), let \( \text{ord}_p(\cdot) \) be the \( p \)-adic additive valuation with \( \text{ord}_p(p) = 1 \) and \( \cdot \mid p \) the \( p \)-adic absolute value which is normalized with \( |p|_p = 1 \). For a real number \( x \), \( \lfloor x \rfloor \) means the greatest integer less than or equal to \( x \). Let \( a \) be a non-zero integer and \( b \) a positive integer. We write \( a \mid b^\infty \) if every prime factor of \( a \) divides \( b \).

Let \( \rho \) be any Dirichlet character. We denote the conductor of \( \rho \) by \( f(\rho) \) and for any prime number \( p \), the \( p \)-primary component of \( \rho \) by \( \rho_p \). Furthermore we set \( \rho_A := \prod_{p \mid A} \rho_p \) for an arbitrary integer \( A \). Here \( p \) runs over all prime divisors of \( A \). We denote by \((\cdot)\) the Kronecker symbol. See [M, p. 82] for a definition of this symbol.
Let $V$ be a finite-dimensional vector space over $C$. We denote the trace of a linear operator $T$ on $V$ by $\text{tr}(T; V)$.

Put $\mu := \text{ord}_2(N)$ and $\nu := \text{ord}_p(N)$ for any odd prime number $p$. Then we decompose $N = 2^\mu M$. Namely, $M$ is the odd part of $N$.

3. Results. Let $\psi$ be a quadratic primitive character with conductor $r$. Then we can express the conductor $r$ as follows:

$$ r = 2^u L, \quad u = 0, 2, \text{and } 3 $$

and $L$ is a squarefree positive odd integer.

We consider the following conditions (1)--(3).

1. $L^2 \mid M$.
2. $L^2 \mid M$ and $\mu \geq 5$, if $f(\chi_2) = 8$.
3. $L^2 \mid M$ and $\mu \geq 6$.

From now on until the end of this paper, we assume the following.

Assumption. We impose the condition (1), (2), or (3) according to $u = 0, 2$, or $3$ respectively.

Now, let $R_\psi$ be the twisting operator of $\psi$:

$$ f = \sum_{n \geq 1} a(n)q^n \mapsto f \mid R_\psi := \sum_{n \geq 1} a(n)\psi(n)q^n, $$

$$(q := \exp(2\pi i z), z \in C, \text{Im } z > 0).$$

Then, from the above conditions (1--3) and the assumption $\psi^2 = 1$, we see that the twisting operator $R_\psi$ fixes the space of cusp forms $S(k + 1/2, N; \chi)$ (cf. [Sh, Lemma 3.6]).

In the case of $k = 1$, we need to make a certain modification. It is well-known that the space $S(3/2, N; \chi)$ contains a subspace $U(N; \chi)$ which corresponds to a space of Eisenstein series via Shimura correspondence and which is generated by theta series of special type (cf. [U2, §0(c)]). Let $V(N; \chi)$ be the orthogonal complement of $U(N; \chi)$ in $S(3/2, N; \chi)$. Then it is also well-known that $V(N; \chi)$ corresponds to a space of cusp forms of weight 2 via Shimura correspondence. Hence we need to consider the subspace $V(N; \chi)$ in place of $S(3/2, N; \chi)$ in the case of $k = 1$. The subspaces $U(N; \chi)$ and $V(N; \chi)$ are fixed by the twisting operator $R_\psi$ (See [U5] for a proof and refer also to [U2, p. 94]). Moreover, the $n^2$-th Hecke operators $T(n^2)$, $(n, N) = 1$, also fix the subspace $V(N; \chi)$ (cf. [U1, p. 508]).

Thus for any positive integer $n$ with $(n, N) = 1$, we can consider the twisted Hecke operator $R_\psi T(n^2)$ on the spaces $S(k + 1/2, N, \chi)$ ($k \geq 2$) and $V(N; \chi)$ ($k = 1$) (cf. [U2, p. 86]).

For the statement of Theorem, we prepare a little more notation.

First we decompose the level $N$ with respect to $L$ as follows:

$$ N = 2^\mu L_0 L_2, \quad L_0 > 0, \quad L_2 > 0, $$

$$ \mu := \text{ord}_2(N), \quad L_0 \mid L^\infty, \quad (L_2, L) = 1. $$

And we put

$$ N_0 := \prod_{p \mid L} 2^2(n_p - 1)/2 + 1. $$

Here $p$ runs over all prime divisors of $L$.

Next, let $A$ be any positive integer. For any odd prime number $p$ and any integers $a, b$ ($0 \leq a \leq \text{ord}_p(A)/2$), we put

$$ \lambda_p(\chi_p, \text{ord}_p(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{\chi_p(b)}{p}\right), & \text{if } 1 \leq a \leq [\text{ord}_p(A) - 1]/2, \\ \chi_p(-b), & \text{if } \text{ord}_p(A) \text{ is even} \\ & \text{and } a = \text{ord}_p(A)/2 \geq 1. \end{cases} $$

And for any integers $a, b$ ($0 \leq a \leq \text{ord}_2(A)/2$), we put

$$ \lambda_2(\chi_2, \text{ord}_2(A); b, a) := \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{if } a = 1, \\ \xi(b)(1 + \left(\frac{\chi}{p}\right)), & \text{if } 2 \leq a \leq [\text{ord}_2(A) - 1]/2, \\ \xi(b)\chi_2(-b), & \text{if } \text{ord}_2(A) \text{ is even} \\ & \text{and } a = \text{ord}_2(A)/2 \geq 2. \end{cases} $$

Here, $\xi(b) := (1 - (\frac{b}{p}))^2$.

Then for any integer $b$ and any square integer $c$, we put

$$ \Lambda_b(\psi, A; b, c) := \prod_{(p, r) = 1} \lambda_p(\chi_p, \text{ord}_p(A); b, \text{ord}_p(c)/2). $$

Here $p$ runs over all prime divisors of $A$ prime to $r$.

Furthermore, let $B$ be a positive integer such that $B \mid r^\infty$ and $(A/B, B) = 1$. For all positive integers $n$ such that $(n, N) = 1$, we define
\[ \Theta_{\psi}[2k, n; A, B, \chi] = \Theta_{\psi}[A, B, \chi] \]
\[ := \sum_{\psi_1, \psi_2 \in \mathbb{Z}[A], N_1 = \mathbb{Z}[A]} \Lambda_\psi(\psi, A; r, n, N_1) \times \text{tr}(W(BN_1)T(n); S(2k, N_1 N_2)), \]
where \( N_1 \) runs over all square divisors of \( A \) which are prime to \( r \) and \( N_2 := A \prod_{p|N_1} |A|_p \).

**Remark.** All the spaces which occur in the definition of \( \Theta_{\psi}[A, B, \chi] \) are contained in the space \( S(2k, A) \).

Finally, let \( \chi_r \) be the \( r \)-primary component of \( \chi \) and \( \chi_r := \prod_{p|N_1} \chi_{p} \), where \( p \) runs over all prime divisors of \( N \) which are prime to \( r \). Then we put
\[ c(k, n; \psi, \chi) = c(\psi, \chi) := \psi(-1)^{k} \chi_{r}(n) \chi_{r}(-r). \]

Under these notations, we can state trace identities of the twisted Hecke operators \( R_{\psi} \tilde{T}(n^2) \).

First we state trace identities for the case of odd conductors.

**Theorem 1.** Let \( k, N, \) and \( \chi \) be the same as above. Suppose that \( \psi \) is a quadratic primitive character defined modulo an odd positive integer \( r \). Hence we assume the condition \( (\ast) \).

For all positive integers \( n \) such that \( (n, N) = 1 \), we have the following trace identities.

(1) Suppose that \( \mu = 2 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, N, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(N; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = c(\psi, \chi) \Theta_{\psi}[N_0, N_2, N_0, \chi]. \]

(2) Suppose that \( 2 \leq \mu \leq 4 \) and furthermore \( f(\chi_2) = 8 \) if \( \mu = 4 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, N, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(N; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = c(\psi, \chi) \Theta_{\psi}[2^{\mu-1}N_0 L_2, N_0, \chi]. \]

(3) Suppose that \( 4 \leq \mu \leq 6 \) and furthermore \( f(\chi_2) \) divides 4 if \( \mu = 4, 6 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, N, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(N; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = 2c(\psi, \chi) \Theta_{\psi}[2^{\mu-2}N_0 L_2, N_0, \chi]. \]

(4) Suppose that \( \mu = 6 \) and \( f(\chi_2) = 8 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, 2^{6}M, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(2^{6}M; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = 4c(\psi, \chi) \times \left\{ \begin{array}{l}
\Theta_{\psi}[2^{3}N_0 L_2, N_0, \chi] \\
- \Theta_{\psi}[2^{2}N_0 L_2, N_0, \chi]\end{array} \right\}. \]

(5) Suppose that \( \mu = 7 \) and \( f(\chi_2) \) divides 4. We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, 2^{7}M, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(2^{7}M; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = 2c(\psi, \chi) \times \left\{ \begin{array}{l}
\Theta_{\psi}[2^{5}N_0 L_2, N_0, \chi] \\
- \Theta_{\psi}[2^{4}N_0 L_2, N_0, \chi]\end{array} \right\}. \]

(6) Suppose that \( \mu = 7 \) and \( f(\chi_2) = 8 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, 2^{7}M, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(2^{7}M; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = 2c(\psi, \chi) \times \left\{ \begin{array}{l}
\Theta_{\psi}[2^{5}N_0 L_2, N_0, \chi] \\
- \Theta_{\psi}[2^{4}N_0 L_2, N_0, \chi] - \Theta_{\psi}[2^{3}N_0 L_2, N_0, \chi] + \Theta_{\psi}[2^{3}N_0 L_2, N_0, \chi] + \Theta_{\psi}[2^{2}N_0 L_2, N_0, \chi]\end{array} \right\}. \]

(7) Suppose that \( \mu \geq 8 \). We have
\[ \begin{align*}
\{ & \text{tr}(R_{\psi} \tilde{T}(n^2); S(k + 1/2, 2^{\mu}M, \chi)) \text{ if } k \geq 2, \\
& \text{tr}(R_{\psi} \tilde{T}(n^2); V(2^{\mu}M; \chi)) \text{ if } k = 1 \}
\end{align*} \]
\[ = 2c(\psi, \chi) \times \left\{ \begin{array}{l}
\Theta_{\psi}[2^{\mu-2}N_0 L_2, N_0, \chi] \\
- \Theta_{\psi}[2^{\mu-3}N_0 L_2, N_0, \chi]\end{array} \right\}. \]
Next, we state trace identities for the case of even conductor.

**Theorem 2.** Let $k$, $N$, and $\chi$ be the same as above. Suppose that $\psi$ is a quadratic primitive character defined modulo an even positive integer $r$. Hence we assume the condition $(\ast 2)$ or $(\ast 3)$ according to $u = 2$ or 3 respectively. 

For all positive integers $n$ such that $(n, N) = 1$, we have the following trace identities.

**Case I.** $(u = 2)$ \( \Rightarrow \psi_2 = (\frac{-1}{n}) \)

(I-1) Suppose that $\mu = 4$ and $f(\chi_2)$ divides 4.

We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^4 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^4 M; \chi)) & \quad \text{if } k = 1
\end{align*}
\]

\[= \chi_2 \left( \frac{1}{Ln} \right) c(\psi, \chi) \Theta_\psi [2^2 N_0 L_2, 2^2 N_0, \chi].\]

(I-2) Suppose that $\mu = 5$ and $f(\chi_2)$ divides 4.

We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & \quad \text{if } k = 1
\end{align*}
\]

\[= \chi_2 \left( \frac{1}{Ln} \right) c(\psi, \chi) \times \{ \Theta_\psi [2^2 N_0 L_2, 2^2 N_0, \chi] - \Theta_\psi [2^2 N_0 L_2, 2^2 N_0, \chi] \}.\]

(I-3) Suppose that $\mu = 5, 6$ and $f(\chi_2)$ = 8. We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & = 0 \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & = 0 \quad \text{if } k = 1.
\end{align*}
\]

(I-4) Suppose that $\mu = 7$ and $f(\chi_2)$ = 8. We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^8 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^8 M; \chi)) & \quad \text{if } k = 1
\end{align*}
\]

\[= (1 - \psi(-1) \left( \frac{-1}{n} \right)) c(\psi, \chi) \times \{ \Theta_\psi [2^5 N_0 L_2, 2^6 N_0, \chi] - \Theta_\psi [2^4 N_0 L_2, 2^4 N_0, \chi] \}.\]

(I-5) Suppose that $\mu \geq 8$, or $\mu = 6, 7$ and $f(\chi_2)$ divides 4.

We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & \quad \text{if } k = 1
\end{align*}
\]

\[= (1 - \psi(-1) \left( \frac{-1}{n} \right)) c(\psi, \chi) \times \Theta_\psi [2^6 - 2 N_0 L_2, 2^5 - 2 N_0, \chi].\]

Here $\hat{\mu}$ is the greatest even integer less than or equal to $\mu$, i.e. $\hat{\mu} = 2[\mu/2]$.

**Case II.** $(u = 3)$ \( \Leftrightarrow \psi_2 = (\frac{\pm 1}{n}) \)

(II-1) Suppose that $\mu = 6, 7$ and $f(\chi_2) = 8$. We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & = 0 \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & = 0 \quad \text{if } k = 1.
\end{align*}
\]

(II-2) Suppose that $\mu \geq 8$, or $\mu = 6, 7$ and $f(\chi_2)$ divides 4. We have

\[
\begin{align*}
\text{tr}(R_\psi \tilde{T}(n^2); S(k + 1/2, 2^6 M, \chi)) & \quad \text{if } k \geq 2 \\
\text{tr}(R_\psi \tilde{T}(n^2); V(2^6 M; \chi)) & \quad \text{if } k = 1
\end{align*}
\]

\[= (1 - \psi(-1) \left( \frac{-1}{n} \right)) c(\psi, \chi) \times \Theta_\psi [2^6 - 2 N_0 L_2, 2^5 - 2 N_0, \chi].\]

Here $\hat{\mu}$ is the greatest odd integer less than or equal to $\mu$, i.e. $\hat{\mu} = 2[\mu/2] + 1$.

4. **Concluding remarks.** We can expect to establish a theory of newforms by using these trace identities. In fact, we established a theory of newforms in the case of level $2^n$. See [U6] for the results.

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