<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>ティトニ</td>
<td>飯塚 勝、富崎 松代</td>
</tr>
<tr>
<td>雑誌名</td>
<td>人間文化研究科年報 第25号</td>
</tr>
<tr>
<td>発行日</td>
<td>2010-03-31</td>
</tr>
<tr>
<td>URL</td>
<td>Nara Women's University Digital Information Repository</td>
</tr>
</tbody>
</table>
Transition probability densities of birth and death processes with finite state space

IIZUKA Masaru* and TOMISAKI Matsuyo**

1 Introduction

Birth and death processes are used to describe stochastic models in various fields. For example, Moran [9] introduced a birth and death process with finite state space as one of fundamental stochastic models in population genetics called Moran model. For birth and death processes with finite state space, Iizuka and Tomisaki [3] considered stochastic processes induced by conditioning on hitting the right boundary point before hitting the left boundary point. To study the behavior of these induced stochastic processes, it is necessary to explore the properties of transition probability densities of birth and death processes with finite state space. In this paper, we present explicit expression of the transition probability densities of birth and death processes with finite state space when the boundary points are absorbing or reflecting.

For \( N \geq 2 \) and \( a_0 < a_1 < a_2 < \cdots < a_N \), let \( \mathcal{D} = [X(t), P_x] \) be a birth and death process with the state space \( \Sigma = \{a_0, a_1, a_2, \ldots, a_N\} \) satisfying the following conditions. For \( a_i \in \Sigma \) and \( \tau_k < t < \tau_{k+1} \), conditional probabilities conditional on \( X(\tau_k) = a_i \) satisfy

\[
\begin{align*}
(1.1) & \quad P_x(X(t) = a_i | X(\tau_k) = a_i) = 1, \\
(1.2) & \quad P_x(X(\tau_{k+1}) = a_{i+1} | X(\tau_k) = a_i) = p_i, \\
(1.3) & \quad P_x(X(\tau_{k+1}) = a_{i-1} | X(\tau_k) = a_i) = q_i, \\
(1.4) & \quad P_x(X(\tau_{k+1}) = a_i | X(\tau_k) = a_i) = 1 - p_i - q_i = r_i.
\end{align*}
\]

Here \( \tau_0 = 0 \) and \( \tau_k = \sum_{i=1}^{k} \epsilon_i \) \((k = 1, 2, \ldots)\) with a sequence \( \{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\} \) of independent copies of \( \epsilon \) which is an exponentially distributed random variable with the mean 1. Further \( 0 \leq p_0 \leq 1, \quad p_N = 0, \quad q_0 = 0, \quad 0 \leq q_N \leq 1, \) and \( p_i > 0, \quad q_i > 0, \quad r_i > 0 \) for \( i = 1, 2, \ldots, N-1 \). The probability measure \( P_x \) is that concentrated at the event \( \{X(0) = x\} \), that is, \( P_x(X(0) = x) = 1 \). The end (boundary) point \( a_0 \) [resp. \( a_N \)] is called to be absorbing or reflecting according to \( p_0 = 0 \) [resp. \( q_N = 0 \)] or \( p_0 > 0 \) [resp. \( q_N > 0 \)].

Such birth and death processes were treated by Feller [2]. He showed that the generator \( \mathcal{L} \) of \( \mathcal{D} \) is given by

\[
(1.5) \quad \mathcal{L}u(a_i) = p_i\{u(a_{i+1}) - u(a_i)\} - q_i\{u(a_i) - u(a_{i-1})\}, \quad i = 1, 2, \ldots, N-1,
\]

for \( u \in D(\mathcal{L}) \), where \( D(\mathcal{L}) \) is the set of all functions \( u \) on \( \Sigma \) such that

\[
\begin{align*}
u(a_0) & = 0 \quad \text{if } a_0 \text{ is absorbing}, \\
\mathcal{L}u(a_0) & = p_0\{u(a_1) - u(a_0)\} \quad \text{if } a_0 \text{ is reflecting}, \\
u(a_N) & = 0 \quad \text{if } a_N \text{ is absorbing},
\end{align*}
\]

*Division of Mathematics, Kyushu Dental College

** School of Interdisciplinary Research of Scientific Phenomena and Information Department of Mathematics and Physics of Fundamental Structures
\[ Lu(a_N) = -q_N\{u(a_N) - u(a_{N-1})\} \quad \text{if } a_N \text{ is reflecting.} \]

The formula (1.5) suggests that \( \mathcal{D} \) can be treated as a one-dimensional generalized diffusion process (ODGDP for brief), by putting \( l_i, i = 1, 2, s \) and \( m \) as follows.

\begin{equation}
(1.6) \quad l_1 = \begin{cases} a_0 & \text{if } p_0 = 0, \\ -\infty & \text{if } p_0 > 0, \end{cases}, \quad l_2 = \begin{cases} a_N & \text{if } q_N = 0, \\ \infty & \text{if } q_N > 0, \end{cases}
\end{equation}

\begin{equation}
(1.7) \quad s(x) = \begin{cases} x - a_0, & l_1 < x \leq a_0, \text{ if } p_0 > 0, \\ x - a_0, & a_0 \leq x < a_1, \\ \frac{a_i - a_0}{p_1p_2\cdots p_i} \cdot x - a_i, & a_i \leq x < a_{i+1}, i = 1, \ldots, N-1, \\ s(a_N) + x - a_N, & a_N \leq x < l_2, \text{ if } q_N > 0, \end{cases}
\end{equation}

\begin{equation}
(1.8) \quad m(x) = \begin{cases} -\infty, & x < l_1, \text{ if } p_0 = 0, \\ -1/p_0, & l_1 \leq x < a_0, \text{ if } p_0 > 0, \\ 0, & a_0 \leq x < a_1, \\ 1/q_1, & a_1 \leq x < a_2, \\ m(a_{i-}) + \frac{p_1p_2\cdots p_{i-1}}{q_1q_2\cdots q_i} x - a_{i-}, & a_i \leq x < a_{i+1}, i = 2, \ldots, N-1, \\ m(a_N-) + \frac{p_1p_2\cdots p_{N-1}}{q_1q_2\cdots q_N} x - a_N, & a_N \leq x < l_2, \text{ if } q_N > 0, \\ \infty, & l_2 \leq x, \text{ if } q_N = 0. \end{cases}
\end{equation}

Let \( \mathcal{G} \) be the one-dimensional generalized diffusion operator (ODGDO for brief) with \((s, m)\), which is explained in the next section. The functions \( s \) and \( m \) are called the scale function and the speed measure function, respectively. It is known that there exists a strong Markov process \( \mathcal{D}^* \) with the generator \( \mathcal{G} \), which is called an ODGDP on \((l_1, l_2)\) (see [4], [12]). In the next section, we observe that \( \mathcal{D} \) can be identified with \( \mathcal{D}^* \). We note that the induced measure \( dm \) is a discrete measure. This is the speed measure for \( \mathcal{D} \).

To make the boundary conditions at the end points \( a_0 \) and \( a_N \) clear, we use \( \mathcal{D}^{IJ} \) and \( P_x^{IJ} \) for \( \mathcal{D} \) and \( P_x \), respectively. Here \( I, J \in \{A, R\} \) and \( I = A \) [resp. \( J = A \)] means that \( a_0 \) [resp. \( a_N \)] is absorbing (i.e. \( p_0 = 0 \) [resp. \( q_N = 0 \)]) and \( I = R \) [resp. \( J = R \)] means that \( a_0 \) [resp. \( a_N \)] is reflecting (i.e. \( p_0 > 0 \) [resp. \( q_N > 0 \)]). It is known that there exists the transition probability density \( p^{IJ}(t, x, y) \) of \( \mathcal{D}^{IJ} \) with respect to the speed measure, that is,

\begin{equation}
(1.9) \quad P_x^{IJ}(X(t) = y) = p^{IJ}(t, x, y)m(\{y\}), \quad t > 0, \ x, y \in \Sigma^*,
\end{equation}

where \( \Sigma^* = \Sigma \cap (l_1, l_2) \) and \( m(\{y\}) = m(y) - m(y^-) \) (see [4], [7]).

The aim of this paper is to give explicit representations of \( p^{IJ}(t, x, y) \), \( I, J \in \{A, R\} \). In the next section we state our main result. We give the proof in Section 3. In Section 4 we consider a simple birth and death process and give representations of its transition probability densities according to boundary conditions.

## 2 Representations of transition probability densities

Let \( \mathcal{D}, \mathcal{D}^{IJ}, s, m, \) etc. be those in the preceding section. We set \( S = (l_1, l_2) \). We simply write \( f(l_1) \) [resp. \( f(l_2) \)] in place of \( f(l_{1+}) \) [resp. \( f(l_{2-}) \)] for a function \( f \) on \( S \), provided
\[ f(l_{1+}) \text{ [resp. } f(l_{2-}) \text{]} \text{ exists. Further, } f^+ \text{ [resp. } f^- \text{]} \text{ stands for the right [resp. left] derivative of } f \text{ with respect to } s \text{ if it exists, that is, } f^+(x) = \lim_{\varepsilon \to 0^+} \frac{f(x+\varepsilon) - f(x)}{s(x+\varepsilon) - s(x)} \text{ [resp. } f^-(x) = \lim_{\varepsilon \to 0^-} \frac{f(x-\varepsilon) - f(x)}{s(x-\varepsilon) - s(x)} \text{].} \]

Let \( D(\mathcal{G}) \) be the space of all bounded continuous functions \( u \) on \( S \) satisfying the following conditions.

\((\mathcal{G}.1)\) There exist a function \( f \) on \( \Sigma^* \) and two constants \( A_1, A_2 \) such that

\[ u(x) = A_1 + A_2(s(x) - s(c)) + \int_{(c,x)} \{s(x) - s(y)\} f(y) \, dm(y), \quad x \in S. \]

\((\mathcal{G}.2)\) For each \( i = 1, 2 \), \( u(l_i) = 0 \) if \( |l_i| < \infty \).

We denote by \( c \) an arbitrarily fixed point of \( \Sigma^* \) throughout this paper. We define the operator \( \mathcal{G} \) by the mapping from \( u \in D(\mathcal{G}) \) to \( f \) appeared in (2.1) and this operator \( \mathcal{G} \) is called the ODGD with \((s, m)\). Let \( \mathcal{D} \) be an ODGD with the generator \( \mathcal{G} \). The birth and death process \( \mathcal{D} \) can be identified with \( \mathcal{D}^* \) (see [2], [4], [12]). Indeed, we can see easily that \( u \in D(\mathcal{G}) \) satisfies the following:

\[ u^+(a_i) = u^-(a_i) = \frac{u(a_{i+1}) - u(a_i)}{(s_{i+1} - s_i)}, \quad i = 0, 1, \ldots, N - 1. \]

\[ u(a_0) = 0 \quad \text{if } p_0 = 0. \]

\[ u(a_N) = 0 \quad \text{if } q_N = 0. \]

\[ u(a_{N+1}) = 0 \quad \text{if } q_{N+1} = 0. \]

Now we turn to \( p^J(t, x, y), I, J \in \{A, R\} \), defined by (1.9). Let \( \varphi_i(x, \lambda), i = 1, 2, \lambda \in \mathbb{C}, x \in S \), be the solutions of the integral equations

\[ \varphi_1(x, \lambda) = 1 + \lambda \int_{[a_*, x]} \{s(x) - s(y)\} \varphi_1(y, \lambda) \, dm(y), \]

\[ \varphi_2(x, \lambda) = s(x) - s(a_*) + \lambda \int_{[a_*, x]} \{s(x) - s(y)\} \varphi_2(y, \lambda) \, dm(y), \]

where \( a_* = a_0 \) if the end point \( a_0 \) is absorbing, and \( a_* \in (-\infty, a_0) \) is fixed arbitrarily if the end point \( a_0 \) is reflecting.

For \( i = 1, 2 \), we denote by \( \lambda_{i,k} \) [resp. \( \lambda_{i,k}^+ \)] the zeros of \( \varphi_i(l_i, -\lambda) \) [resp. \( \varphi_i^+(a_N, -\lambda) \)], that is, \( \varphi_i(l_i, -\lambda_{i,k}) = 0 \) [resp. \( \varphi_i^+(a_N, -\lambda_{i,k}^+) = 0 \)], where \( \varphi_i(l_i, -\lambda) = \lim_{x \to l_i} \varphi_i(x, -\lambda) \) if \( a_N = l_i \) [resp. \( \varphi_i^+(a_N, -\lambda) = \lim_{x \to a_N} \varphi_i(a_N + \varepsilon, -\lambda) - \varphi_i(a_N, -\lambda)\) if \( a_N < l_i \)]. For \( I, J \in \{A, R\} \), we consider the following \( N^{I,J}, \lambda_{I,k}^{I,J}, \psi_{I,k}^{I,J} \) and \( \kappa_{I,k}^{I,J} \).

\[ N^{AA} = N - 1, \quad \lambda_{k}^{AA} = \lambda_{2,k}, \quad \psi_{k}^{AA}(x) = \varphi_2(x, -\lambda_{k}^{AA}), \]

\[ \kappa_{k}^{AA} = \varphi_1(l_2, -\lambda_{k}^{AA}) \frac{\partial}{\partial \lambda} \varphi_2(l_2, -\lambda) \bigg|_{\lambda = \lambda_{k}^{AA}}. \]

\[ N^{AR} = N, \quad \lambda_{k}^{AR} = \lambda_{2,k}^+, \quad \psi_{k}^{AR}(x) = \varphi_2(x, -\lambda_{k}^{AR}), \]

\[ \kappa_{k}^{AR} = 0. \]
Now we state our main result.

**Proposition 2.1**  It holds true that

\[ p_{ij}(t, x, y) = \sum_{k=1}^{N_{ij}} \exp\{-\lambda_k^j t\} \psi_k^j(x) \psi_k^j(y) \kappa_k^{ij}, \]

for \( t > 0, \ x, y \in \Sigma, \) and \( i, j \in \{A, R\} \).

## 3 Proof of Proposition 2.1

Let \( D^ij, S, s, m, \) etc. be those given in the preceding sections. We set \( m_t = m({a_i}), \ s_t = s({a_i}), \ i = 0, 1, \cdots, N \). We note that \( m_0 = \infty \) [resp. \( m_N = \infty \)] if the end point \( a_0 \) [resp. \( a_N \)] is absorbing. Let \( \varphi_i(x, \lambda), i = 1, 2, \lambda \in \mathbb{C}, \ x \in S, \) be the solutions of the integral equations (2.2) and (2.3). For \( \alpha > 0, \) we set

\[ h(\alpha) = \lim_{\varepsilon \to 0} \varphi_2(x, \alpha) \varphi_1(x, \alpha), \]

The function \( h(\alpha) \) can be analytically continued to \( \mathbb{C} \setminus (-\infty, 0]. \) The spectral measures \( \sigma^ij, i, j \in \{A, R\} \) are defined by

\[ \sigma^A_{ij}([\lambda_1, \lambda_2]) = -\lim_{\varepsilon \to \infty} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda - \sqrt{\lambda - \lambda_1 \varepsilon}} d\lambda, \]

\[ \sigma^R_{ij}([\lambda_1, \lambda_2]) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda - \sqrt{\lambda - \lambda_1 \varepsilon}} d\lambda, \]

for all continuity points \( \lambda_1 \) and \( \lambda_2 \) of \( \sigma^ij (\lambda_1 < \lambda_2). \) For each \( i, j \in \{A, R\}, \) the transition probability density \( p^ij(t, x, y) \) is defined as follows (see [4], [7], [13], see also [8]).

\[ p^A_{ij}(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \varphi_2(x, -\lambda) \varphi_1(y, -\lambda) \sigma^A_{ij}(d\lambda), \]

\[ p^R_{ij}(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \varphi_1(x, -\lambda) \varphi_1(y, -\lambda) \sigma^R_{ij}(d\lambda), \]

for \( i \in \{A, R\}, \ t > 0, \ x, y \in \Sigma. \) We note that \( \varphi_i(x, -\lambda) \) and \( \sigma^R_{ij}(d\lambda) \) do not depend on \( a_* \) in the integral equations (2.2) and (2.3).

We divide the proof of Proposition 2.1 into four cases. We denote by \( \varphi^+_i(x, \lambda) \) the right derivative of \( \varphi_i(x, \lambda) \) with respect to \( s(x). \)
3.1 Representation of $p^{AA}(t, x, y)$

We consider the case that both of the end points $a_0$ and $a_N$ are absorbing. Then $p_0 = q_N = 0$ and $l_1 = a_0, l_2 = a_N$. By means of (2.2) and (2.3) with $a_* = a_0$, we obtain the following equalities.

For $a_0 < x < a_1$,

\[(3.6) \quad \varphi_1(x, \lambda) = 1, \quad \varphi_2(x, \lambda) = s(x).\]

For $a_k < x < a_{k+1}, k = 1, 2, \ldots, N-1$,

\[(3.7) \quad \varphi_1(x, \lambda) = 1 + \lambda \sum_{1 \leq i \leq k} \{s(x) - s_i\}m_i + \sum_{j=2}^{k} \lambda^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} \{s(x) - s_{i_j}\}(s_{i_j} - s_{i_{j-1}})(s_{i_{j-1}} - s_{i_{j-1}})m_{i_j}m_{i_{j-1}} \ldots m_{i_1},\]

\[(3.8) \quad \varphi_2(x, \lambda) = s(x) + \lambda \sum_{1 \leq i \leq k} \{s(x) - s_i\}m_i + \sum_{j=2}^{k} \lambda^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} \{s(x) - s_{i_j}\}(s_{i_j} - s_{i_{j-1}})(s_{i_{j-1}} - s_{i_{j-1}})s_{i_1}m_{i_j}m_{i_{j-1}} \ldots m_{i_1}.\]

By means of (3.7) and (3.8), $\varphi_1(l_2, \lambda) = \lim_{x \to l_2} \varphi_1(x, \lambda), i = 1, 2, \ldots, N-1$, are polynomials of degree $N-1$. By virtue of (A.5) with $k = N$ in Appendix, we obtain that

\[(3.9) \quad 0 < \lambda_{1,1} < \lambda_{2,1} < \lambda_{1,2} < \lambda_{2,2} < \ldots < \lambda_{1,N-1} < \lambda_{2,N-1}.\]

Since (3.1) is reduced to $h(\alpha) = \varphi_2(l_2, \alpha)/\varphi_1(l_2, \alpha)$, by virtue of (3.9) and Proposition A of [10], we see that $\kappa^{AA}_{k} > 0, k = 1, 2, \ldots, N-1$, and $\sigma^{AA}$ is a measure having mass $\kappa^{AA}_{k}$ only at $\lambda^{AA}_{k}$, that is,

\[\sigma^{AA}(d\lambda) = \sum_{k=1}^{N-1} \kappa^{AA}_{k} \delta_{\lambda^{AA}_{k}}(d\lambda),\]

where $\delta_a$ is the unit measure at $a$. Combining this with (3.4), we obtain

\[(3.10) \quad p^{AA}(t, x, y) = \sum_{1 \leq k \leq N-1} \exp\{-\lambda^{AA}_{k}t\} \psi_k^{AA}(x) \psi_k^{AA}(y) \kappa^{AA}_{k},\]

for $t > 0, x, y \in \Sigma$. This shows (2.4) with $I = J = A$.

3.2 Representation of $p^{AR}(t, x, y)$

We consider the case that the end point $a_0$ is absorbing and the end point $a_N$ is reflecting. Then $p_0 = q_N > 0$ and $l_1 = a_0, l_2 = \infty$. By means of (2.2) and (2.3) with $a_* = a_0$, we obtain (3.6), (3.7) and (3.8) with $k = 1, 2, \ldots, N$, where $a_{N+1}$ is read as $l_2$. We note the following.

For $a_0 < x < a_1$,

\[(3.11) \quad \varphi_1^+(x, \lambda) = 0, \quad \varphi_2^+(x, \lambda) = 1.\]
For $a_k \leq x < a_{k+1}$, $k = 1, 2, \ldots, N$,

\begin{equation}
\phi_1^+(x, \lambda) = \lambda \sum_{1 \leq i \leq k} m_i
+ \sum_{j=2}^{k} \lambda^j \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} (s_{i_j} - s_{i_{j-1}}) \cdots (s_{i_2} - s_{i_1}) m_{i_j} m_{i_{j-1}} \cdots m_{i_1},
\end{equation}

\begin{equation}
\phi_2^+(x, \lambda) = 1 + \lambda \sum_{1 \leq i \leq k} s_i m_i
+ \sum_{j=1}^{k} \lambda^j \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} (s_{i_j} - s_{i_{j-1}}) \cdots (s_{i_2} - s_{i_1}) s_{i_j} m_{i_j} m_{i_{j-1}} \cdots m_{i_1}.
\end{equation}

We see that $\phi_i^+(a_N, -\lambda)$, $i = 1, 2$, are polynomials of degree $N$. By virtue of (A.9) with $k = N$, we obtain that

\begin{equation}
0 = \lambda_{1,1}^+ < \lambda_{2,1}^+ < \lambda_{1,2}^+ < \lambda_{2,2}^+ < \cdots < \lambda_{1,N-1}^+ < \lambda_{2,N-1}^+ < \lambda_{1,N}^+ < \lambda_{2,N}^+.
\end{equation}

Note that (3.1) is reduced to $h(\alpha) = \phi_2^+(a_N, \alpha)/\phi_1^+(a_N, \alpha)$. By virtue of (3.14) and Proposition A of [10], we see that $\kappa_k^{AR} > 0$, $k = 1, 2, \ldots, N$, and $\sigma^{AR}$ is a measure having mass $\kappa_k^{AR}$ only at $\lambda_k^{RA}$, that is,

\begin{equation}
\sigma^{AR}(d\lambda) = \sum_{k=1}^{N} \kappa_k^{AR} \delta_{\lambda_k^{RA}}(d\lambda).
\end{equation}

Combining this with (3.4), we obtain

\begin{equation}
p^{AR}(t, x, y) = \sum_{1 \leq k \leq N} \exp\{-\lambda_k^{AR}t\} \psi_k^{AR}(x) \psi_k^{AR}(y) \kappa_k^{AR},
\end{equation}

for $t > 0$, $x, y \in \Sigma$. Thus we get Proposition 2.1 with $I = A$, $J = R$.

### 3.3 Representation of $p^{RA}(t, x, y)$

We consider the case that the end point $a_0$ is reflecting and the end point $a_N$ is absorbing. Then $p_0 > 0$, $q_N = 0$ and $l_1 = -\infty$, $l_2 = a_N$. In this case, we fix a point $a_* \in (l_1, a_0)$ arbitrarily and consider (2.2) and (2.3). We set $s_0 = s(a_*)$. Then we obtain the following.

For $l_1 < x < a_0$,

\begin{equation}
\phi_1(x, \lambda) = 1, \quad \phi_2(x, \lambda) = s(x) - s_0.
\end{equation}

For $a_k \leq x < a_{k+1}$, $k = 0, 1, \ldots, N - 1$,

\begin{equation}
\phi_1(x, \lambda) = 1 + \lambda \sum_{0 \leq i \leq k} \{s(x) - s_i\} m_i
+ \sum_{j=2}^{k+1} \lambda^j \sum_{0 \leq i_1 < i_2 < \cdots < i_j \leq k} \{s(x) - s_{i_1}\} (s_{i_1} - s_{i_{j-1}}) \cdots (s_{i_2} - s_{i_1}) m_{i_j} m_{i_{j-1}} \cdots m_{i_1},
\end{equation}

\begin{equation}
\phi_2(x, \lambda) = 1 + \lambda \sum_{0 \leq i \leq k} s_i m_i
+ \sum_{j=1}^{k+1} \lambda^j \sum_{0 \leq i_1 < i_2 < \cdots < i_j \leq k} \{s(x) - s_{i_1}\} (s_{i_1} - s_{i_{j-1}}) \cdots (s_{i_2} - s_{i_1}) s_{i_j} m_{i_j} m_{i_{j-1}} \cdots m_{i_1}.
\end{equation}
\[ \varphi_2(x, \lambda) = s(x) - s_* + \lambda \sum_{0 \leq i \leq k} \{s(x) - s_i\}(s_i - s_*)m_i \]

\[ + \sum_{j=2}^{k+1} \lambda^j \sum_{0 \leq i_1 < i_2 < \cdots < i_j \leq k} \{s(x) - s_{i_j}\}(s_{i_j} - s_{i_{j-1}}) \cdots \times (s_{i_2} - s_{i_1})(s_{i_1} - s_*)m_{i_1}m_{i_{j-1}} \cdots m_{i_1}. \]

By means of (3.17) and (3.18), \( \varphi_i(t_2, -\lambda) = \lim_{x \to t_2} \varphi_2(x, -\lambda), \ i = 1, 2, \) are polynomials of degree \( N. \) By means of (B.1) with \( k = N \) in Appendix, we obtain that

\[ 0 < \lambda_{1,1} < \lambda_{2,1} < \lambda_{1,2} < \lambda_{2,2} < \cdots < \lambda_{1,N-1} < \lambda_{2,N-1} < \lambda_{1,N} < \lambda_{2,N}. \]

Note that (3.1) is reduced to \( h(\alpha) = \varphi_2(t_2, \alpha)/\varphi_1(t_2, \alpha). \) By virtue of (3.19) and Proposition A of [10], we see that \( \kappa_{k,R} > 0, \) \( k = 1, 2, \ldots, N, \) and \( \sigma_{R}^{RA} \) is a measure having mass \( \kappa_{k,R}^{RA} \) only at \( \lambda_{k,R}^{RA}, \) that is,

\[ \sigma_{R}^{RA}(d\lambda) = \sum_{k=1}^{N} \kappa_{k,R}^{RA} \delta_{\lambda_{k,R}^{RA}}(d\lambda). \]

Combining this with (3.5), we obtain

\[ p_{R}^{RA}(t, x, y) = \sum_{1 \leq k \leq N} \exp\{-\lambda_{k,R}^{RA}\} \psi_{k,R}^{RA}(x)\psi_{k,R}^{RA}(y)\kappa_{k,R}^{RA}, \]

for \( t > 0, \ x, y \in \Sigma. \) This shows (2.4) with \( I = R, \ J = A. \)

### 3.4 Representation of \( p_{RR}^{RA}(t, x, y) \)

We consider the case that both of the end points \( a_0 \) and \( a_N \) are reflecting. Then \( p_0 > 0, \ q_N > 0 \) and \( l_1 = -\infty, \ l_2 = \infty. \) In this case, we note that (3.16), (3.17) and (3.18) are valid for \( k = 0, 1, \ldots, N, \) where \( a_{N+1} \) is read as \( a_{N+1} = l_2. \) Therefore we obtain the following.

For \( l_1 < x < a_0, \)

\[ \varphi_1^{+}(x, \lambda) = 0, \quad \varphi_2^{+}(x, \lambda) = 1. \]

For \( a_k \leq x < a_{k+1}, \ k = 0, 1, \ldots, N, \)

\[ \varphi_1^{+}(x, \lambda) = \lambda \sum_{0 \leq i \leq k} m_i \]

\[ + \sum_{j=2}^{k+1} \lambda^j \sum_{0 \leq i_1 < i_2 < \cdots < i_j \leq k} (s_{i_j} - s_{i_{j-1}}) \cdots (s_{i_2} - s_{i_1})m_{i_1}m_{i_{j-1}} \cdots m_{i_1}, \]

\[ \varphi_2^{+}(x, \lambda) = 1 + \lambda \sum_{0 \leq i \leq k} (s_i - s_*)m_i \]

\[ + \sum_{j=2}^{k+1} \lambda^j \sum_{0 \leq i_1 < i_2 < \cdots < i_j \leq k} (s_{i_j} - s_{i_{j-1}})(s_{i_2} - s_{i_1})(s_{i_1} - s_*)m_{i_1}m_{i_{j-1}} \cdots m_{i_1}. \]
These show that $\varphi^+_i(a_N, -\lambda)$, $i = 1, 2$, are polynomials of degree $N + 1$. Since $\lambda^+_k$ are the zeros of polynomials $\varphi^+_i(a_N, -\lambda)$, $i = 1, 2$, by means of (B.5) with $k = N$ we obtain that

$$0 = \lambda^+_1 < \lambda^+_2 < \lambda^+_3 < \cdots < \lambda^+_N < \lambda^+_1 < \lambda^+_2 < \cdots < \lambda^+_N.$$  

Note that (3.1) is reduced to $h(\alpha) = \varphi^+_2(a_N, \alpha)/\varphi^+_1(a_N, \alpha)$. By virtue of (3.24) and Proposition A of [10], we see that $\kappa^R_k > 0$, $k = 1, 2, \ldots, N + 1$, and $\sigma^R$ is a measure having mass $\kappa^R_k$ only at $\lambda^+_k$, that is,

$$\sigma^R(d\lambda) = \sum_{k=1}^{N+1} \kappa^R_k \delta_{\lambda^+_k}(d\lambda).$$

Combining this with (3.5), we obtain

$$p^R(t, x, y) = \sum_{1 \leq k \leq N+1} \exp{-\lambda^+_k t} \psi^R_k(x) \psi^R_k(y) \kappa^R_k,$$

for $t > 0$, $x, y \in \Sigma$. Therefore we have (2.4) with $I = J = R$.

## 4 Examples

In this section, we consider a simple birth and death process. Let $\tau_0 = 0$ and $\tau_k$ ($k = 1, 2, \ldots$) be a sequence of random times introduced in Section 1.

Let $N = 2$, $a_i = i$ ($i = 0, 1, 2$), $q_0 = 0$, $p_2 = 0$ and $p_1 = q_1 = 1/2$. The transition law of this birth and death process $\mathbb{D}$ satisfies

$$P_x(X(\tau_{k+1}) = 0|X(\tau_k) = 1) = P_x(X(\tau_{k+1}) = 2|X(\tau_k) = 1) = 1/2.$$

### 4.1 Case that the end points 0 and 2 are absorbing

We first consider $\mathbb{D}^{AA}$, that is,

$$P_x(X(\tau_{k+1}) = 0|X(\tau_k) = 0) = 1, \quad P_x(X(\tau_{k+1}) = 2|X(\tau_k) = 2) = 1.$$

Then $l_1 = 0$, $l_2 = 2$, (1.7) and (1.8) are reduced to

$$s(x) = x, \quad m(x) = \begin{cases} -\infty, & x < 0, \\ 0, & 0 \leq x < 1, \\ 2, & 1 \leq x < 2, \\ \infty, & 2 \leq x. \end{cases}$$

Following Proposition 2.1, we obtain that

$$P^A_x(X(t) = y) = p^A(x, y|m(y)), \quad p^A(t, x, y) = \frac{1}{2} e^{-t} \chi_{(1)}(x) \chi_{(1)}(y),$$

where $\chi_{\Lambda}(\xi) = 1$ [resp. $\chi_{\Lambda}(\xi) = 0$] if $\xi \in \Lambda$ [resp. $\xi \notin \Lambda$].
4.2 Case that the end point 0 is reflecting and the end point 2 is absorbing

We consider $\mathcal{D}^R$ with $P_0 = 1$, that is,
\[ P_x(X(T_{k+1}) = 1|X(T_k) = 0) = 1, \quad P_x(X(T_{k+1}) = 2|X(T_k) = 2) = 1. \]

Then $l_1 = -\infty$, $l_2 = 2$, (1.7) and (1.8) are reduced to
\[ s(x) = x, \quad m(x) = \begin{cases} -1, & x < 0, \\ 0, & 0 \leq x < 1, \\ 2, & 1 \leq x < 2, \\ \infty, & 2 \leq x. \end{cases} \]

Following Proposition 2.1, we obtain that
\[ P^R(X(t) = y) = P^R(t, x, y)m(\{y\}), \]
\[ P^R(t, x, y) = 2^{-(x+y)/2}X^{(0,1)}(x)X^{(0,1)}(y) \left\{ e^{-\frac{(2-2t)/2}{2}} + (-1)^{x+y}e^{-\frac{(2+2t)/2}{2}} \right\}. \]

4.3 Case that the end point 0 is absorbing and the end point 2 is reflecting

We next consider $\mathcal{D}^A$ with $q_2 = 1$, that is,
\[ P_x(X(T_{k+1}) = 0|X(T_k) = 0) = 1, \quad P_x(X(T_{k+1}) = 1|X(T_k) = 2) = 1. \]

Then $l_1 = 0$, $l_2 = \infty$, (1.7) and (1.8) are reduced to
\[ s(x) = x, \quad m(x) = \begin{cases} -\infty, & x < 0, \\ 0, & 0 \leq x < 1, \\ 2, & 1 \leq x < 2, \\ 3, & 2 \leq x. \end{cases} \]

Following Proposition 2.1, we obtain that
\[ P^A(X(t) = y) = P^A(t, x, y)m(\{y\}), \]
\[ P^A(t, x, y) = 2^{(x+y-6)/2}X^{(1,2)}(x)X^{(1,2)}(y) \left\{ e^{-\frac{(2-2t)/2}{2}} + (-1)^{x+y}e^{-\frac{(2+2t)/2}{2}} \right\}. \]

Since $P^A(t, x, y) = P^R(t, 2-x, 2-y)$, (4.2) also follows from (4.1).

4.4 Case that the end points 0 and 2 are reflecting

We consider $\mathcal{D}^{RR}$ with $P_0 = q_2 = 1$, that is,
\[ P_x(X(T_{k+1}) = 1|X(T_k) = 0) = 1, \quad P_x(X(T_{k+1}) = 1|X(T_k) = 2) = 1. \]

Then $l_1 = -\infty$, $l_2 = \infty$, (1.7) and (1.8) are reduced to
\[ s(x) = x, \quad m(x) = \begin{cases} -1, & x < 0, \\ 0, & 0 \leq x < 1, \\ 2, & 1 \leq x < 2, \\ 3, & 2 \leq x. \end{cases} \]
Following Proposition 2.1, we obtain that
\[
P^R_X(X(t) = y) = p^R(t, x, y)m\{y\},
\]
\[
p^R(t, x, y) = \frac{1}{4} \left\{ 1 + (1 - x)(1 - y)e^{-t} + (-1)^{x+y}e^{-2t} \right\}.
\]

**Appendix**

Let \( S, m, s, \) etc. be those given in Sections 1 and 2. Let \( \varphi_i(x, \lambda), i = 1, 2, \lambda \in \mathbb{C}, x \in S, \) be the solutions of the integral equations (2.2) and (2.3). The aim of this section is to study zeros of \( \varphi_i(x, -\lambda) \) and \( \varphi_i^+(x, -\lambda), i = 1, 2, \) as functions of \( \lambda \in \mathbb{R}. \) We denote by \( \lambda_{i,j}(x) \) and \( \lambda_{i,j}^+(x), j = 1, 2, \cdots, \) zeros of \( \varphi_i(x, -\lambda) \) and \( \varphi_i^+(x, -\lambda), \) respectively. In particular, \( \lambda_{i,j}(a_N), j = 1, 2, \cdots, \) stand for zeros of \( \varphi_i(l_2, -\lambda) = \lim_{\tau \to 0} \varphi_i(x, -\lambda) \) if the end point \( a_N \) is absorbing (and hence \( a_N = l_2). \)

Let \( x \in S. \) Then, by virtue of (2.2) and (2.3),
\[
\varphi_1(x, -\lambda) \geq 1, \quad \varphi_2(x, -\lambda) \geq s(x), \quad \varphi_2^+(x, -\lambda) \geq 1, \quad \text{for } \lambda \leq 0.
\]
Therefore \( \lambda_{i,j}(x) > 0, i = 1, 2, \) and \( \lambda_{2,j}^+(x) > 0, j = 1, 2, \cdots, \) if they exist. We also note that
\[
\varphi_1^+(x, -\lambda) = -\lambda \int_{[x,s]} \varphi_1(y, -\lambda) dm(y).
\]
Therefore we may set \( \lambda_{1,j}^+(x) = 0 \) for \( x \geq a_1. \)

In the following we write \( \lambda_{i,j,k} = \lambda_{i,j}(a_k) \) and \( \lambda_{i,j,k}^+ = \lambda_{i,j}^+(a_k). \)

**A  Case that the end point \( a_0 \) is absorbing**

We consider the case that the end point \( a_0 \) is absorbing, and hence \( \varphi_i(x, \lambda), i = 1, 2, \) are solutions of (2.2) and (2.3) with \( a_* = a_0. \) If the end point \( a_N \) is absorbing, we have (3.7), (3.8), (3.12) and (3.13) for \( k = 1, 2, \cdots, N - 1. \) If the end point \( a_N \) is reflecting, we have (3.7), (3.8), (3.12) and (3.13) for \( k = 1, 2, \cdots, N, \) where \( a_{N+1} \) is read as \( l_2 = \infty. \)

We gather some formulas needed below. It is obvious that
\[
(A.1) \quad \varphi_1(x, -\lambda) - \varphi_2^+(x, -\lambda) - \varphi_2(x, -\lambda) = 1, \quad x \in S, \lambda \in \mathbb{R}.
\]
By virtue of definition of \( \varphi_1^+(x, -\lambda), \)
\[
(A.2) \quad \varphi_1(x, -\lambda) = \varphi_1^+(a_{k-1}, -\lambda)\{s(x) - s_{k-1}\} + \varphi_1(a_{k-1}, -\lambda),
\]
for \( a_{k-1} \leq x \leq a_k \) and \( i = 1, 2. \) We denote \( \varphi_i'(x, -\lambda) = \frac{\partial}{\partial \lambda} \varphi_i(x, -\lambda) \) and \( \varphi_i^{'+}(x, -\lambda) = \frac{\partial}{\partial \lambda} \varphi_i^+(x, -\lambda). \) The following equalities are proved in the same way as those in Section 2.1 of [6] (see also (3.1), (3.2) in [11]).
\[
(A.3) \quad \varphi_i'(x, -\lambda) = \sum_{k,a_k \leq x} \{ \varphi_1(x, -\lambda)\varphi_2(a_k, -\lambda) - \varphi_2(x, -\lambda)\varphi_1(a_k, -\lambda) \} \varphi_i(a_k, -\lambda)m_k,
\]
\[
(A.4) \quad \varphi_i^{'+}(x, -\lambda) = \sum_{k,a_k \leq x} \{ \varphi_1^+(x, -\lambda)\varphi_2(a_k, -\lambda) - \varphi_2^+(x, -\lambda)\varphi_1(a_k, -\lambda) \} \varphi_i(a_k, -\lambda)m_k,
\]
for \( i = 1, 2 \) and \( x \in S. \)
Proposition A.1  Let $i = 1, 2$ and $k = 2, \ldots, N$. Then $\varphi_i(a_k, -\lambda)$ [resp. $\varphi_i^+(a_{k-1}, -\lambda)$] has $k-1$ zeros $\lambda_{i,j,k}$ [resp. $\lambda_{i,j,k-1}^+$] $\left( j = 1, 2, \ldots, k-1 \right)$ such that

\begin{align*}
(A.5) & \quad 0 < \lambda_{1,1,k} < \lambda_{1,2,k} < \lambda_{1,2,k} < \cdots < \lambda_{1,k-1,k} < \lambda_{2,k-1,k}, \\
(A.6) & \quad 0 = \lambda_{1,1,k-1}^+ < \lambda_{1,1,k}^+ < \lambda_{1,2,k-1}^+ < \cdots < \lambda_{1,k-1,k-1}^+ < \lambda_{1,k-1,k}^+, \\
(A.7) & \quad 0 < \lambda_{2,1,k-1}^+ < \lambda_{2,1,k} < \lambda_{2,2,k-1} < \lambda_{2,2,k} < \cdots < \lambda_{2,k-1,k-1}^+ < \lambda_{2,k-1,k}^+.
\end{align*}

Further

\begin{align*}
(A.8) & \quad 0 = \lambda_{1,1,1} < \lambda_{2,1,1}^+,
\end{align*}

and it holds true that

\begin{align*}
(A.9) & \quad 0 = \lambda_{1,1,1} < \lambda_{1,1,2} < \lambda_{2,1,1} < \lambda_{2,1,2} < \cdots < \lambda_{2,k-1,k-1} < \lambda_{2,k-1,k}, \\
(A.10) & \quad 0 = \lambda_{1,1,1} < \lambda_{1,1,2} < \lambda_{1,2,1} < \cdots < \lambda_{1,k-1,k-1} < \lambda_{1,k-1,k}, \\
(A.11) & \quad 0 < \lambda_{2,1,1}^+ < \lambda_{2,1,2} < \lambda_{2,2,1} < \lambda_{2,2,2} < \cdots < \lambda_{2,k-1,k-1}^+.
\end{align*}

for $k = 2, 3, \ldots, N - 1$. In particular, if the end point $a_N$ is reflecting, $\varphi_i^+(a_N, -\lambda)$, $i = 1, 2$, have $N$ zeros $\lambda_{i,j,N}^+$, and (A.9), (A.10) and (A.11) hold for $k = N$.

**Proof** It follows from (3.7), (3.8), (3.12) and (3.13) that

\begin{align*}
\varphi_1(a_2, -\lambda) &= 1 - \lambda(s_2 - s_1)m_1, \quad \varphi_1^+(a_1, -\lambda) = -\lambda m_1, \\
\varphi_2(a_2, -\lambda) &= s_2 - \lambda(s_2 - s_1)s_1 m_1, \quad \varphi_2^+(a_1, -\lambda) = 1 - \lambda s_1 m_1.
\end{align*}

Each of $\varphi_i(a_2, -\lambda)$ and $\varphi_i^+(a_1, -\lambda)$, $i = 1, 2$, has one zero such that

\begin{align*}
\lambda_{1,1,2} &= 1/(s_2 - s_1)m_1, \quad \lambda_{1,1,1}^+ = 0, \\
\lambda_{2,1,2} &= s_2/(s_2 - s_1)s_1 m_1, \quad \lambda_{2,1,1}^+ = 1/s_1 m_1,
\end{align*}

which show (A.5), (A.6), (A.7) with $k = 2$, and (A.8). By using (A.5) with $k = 2$, we get

\begin{align*}
(A.12) & \quad \varphi_1(a_2, -\lambda_{2,1,2}) < 0, \quad \varphi_2(a_2, -\lambda_{1,1,2}) > 0.
\end{align*}

By means of (A.1),

\begin{align*}
\varphi_1(a_2, -\lambda_{2,1,2}) &\varphi_2^+(a_2, -\lambda_{2,1,2}) = 1, \quad -\varphi_1^+(a_2, -\lambda_{1,1,2})\varphi_2(a_2, -\lambda_{1,1,2}) = 1.
\end{align*}

These coupled with (A.12) imply

\begin{align*}
(A.13) & \quad \varphi_1^+(a_2, -\lambda_{1,1,2}) < 0, \quad \varphi_2^+(a_2, -\lambda_{2,1,2}) < 0.
\end{align*}

The equalities (3.12), (3.13) and (A.13) imply that, for $i = 1, 2$, $\varphi_i^+(a_2, -\lambda)$ has two zeros $\lambda_{i,j,2}^+$, $j = 1, 2$, satisfying (A.10) and (A.11) with $k = 2$. We note that $(-1)^j\varphi_i^+(a_2, -\lambda_{i,j,2}^+) > 0$, $j = 1, 2$. Then we obtain by (A.4),

\begin{align*}
(A.14) & \quad (-1)^j\varphi_2^+(a_2, -\lambda_{i,j,2}^+) < 0, \quad j = 1, 2,
\end{align*}

from which (A.9) with $k = 2$ follows. Thus each of $\varphi_i(a_2, -\lambda)$, $\varphi_i^+(a_1, -\lambda)$, $i = 1, 2$, has one zero satisfying (A.5), (A.6), (A.7) with $k = 2$, and (A.8). Further each of $\varphi_i^+(a_2, -\lambda)$, $i = 1, 2$, has two zeros satisfying (A.9), (A.10) and (A.11) with $k = 2$. -281-
Let \( i = 1, 2 \) and \( l \geq 2 \). Suppose that each of \( \varphi_i(a_{l-1}, -\lambda) \), \( \varphi_i^+(a_{l-1}, -\lambda) \), has \( l - 1 \) zeros satisfying (A.5), (A.6) and (A.7) with \( k = l \), and \( \varphi_i^+(a_{l-1}, -\lambda) \) has \( l \) zeros satisfying (A.9), (A.10) and (A.11) with \( k = l \). By (A.10) and (A.11) with \( k = l \), we see that

\[
(-1)^j \varphi_i^+(a_l, -\lambda_{l,j,l}) > 0, \quad j = 1, 2, \ldots, l - 1. \tag{A.15}
\]

Combining this with (A.2), we see that

\[
(-1)^j \varphi_i(a_{l+1}, -\lambda_{l,j,l+1}) = (-1)^j \varphi_i^+(a_l, -\lambda_{l,j,l})(s_{l+1} - s_l) > 0, \quad j = 1, 2, \ldots, l - 1. \tag{A.16}
\]

The equalities (3.7), (3.8) and (A.15) imply that \( \varphi_i(a_{l+1}, -\lambda) \) has \( l \) zeros \( \lambda_{l,j,l+1} \), \( j = 1, 2, \ldots, l \), satisfying

\[
0 < \lambda_{l,j,l+1} < \lambda_{l+1,j,l} < \lambda_{l+2,j,l} < \cdots < \lambda_{l,l-1,j} < \lambda_{l,l,j+1}, \tag{A.17}
\]

and hence \( (-1)^j \varphi_i(a_{l+1}, -\lambda_{l,j,l+1}) > 0, \quad j = 1, 2, \ldots, l \). Combining this with (A.3), we get

\[
(-1)^j \varphi_i(a_{l+1}, -\lambda_{l,j,l+1}) > 0, \quad (-1)^j \varphi_2(a_{l+1}, -\lambda_{l,j,l+1}) < 0, \quad j = 1, 2, \ldots, l. \tag{A.18}
\]

Therefore we obtain (A.5) with \( l + 1 \) in place of \( k \). It follows from (A.2) that

\[
(-1)^j \varphi_i^+(a_l, -\lambda_{l,j,l+1})(s_{l+1} - s_l) + \varphi_i(a_l, -\lambda_{l,j,l+1}) = 0, \quad j = 1, 2, \ldots, l. \tag{A.19}
\]

This coupled with (3.12) and (3.13) implies that \( \varphi_i^+(a_{l+1}, -\lambda) \) has \( l + 1 \) zeros \( \lambda_{l,j,k+1}^+ \), \( j = 1, 2, \ldots, l + 1 \), for which \( (A.10) \) and \( (A.11) \) hold true with \( l + 1 \) in place of \( k \). Since \( (-1)^j \varphi_i^+(a_{l+1}, -\lambda_{l,j,l+1}) > 0, \quad l = 1, 2, \ldots, l + 1 \), we obtain by (A.4)

\[
(-1)^j \varphi_2^+(a_{l+1}, -\lambda_{l,j,l+1}) > 0, \quad j = 1, 2, \ldots, l + 1, \tag{B.1}
\]

which implies that (A.9) holds with \( k \) replaced by \( l + 1 \). Thus we obtain the statement of the proposition for \( k = l + 1 \).}

\[ \square \]

B Case that the end point \( a_0 \) is reflecting

We consider the case that the end point \( a_0 \) is reflecting. Therefore we have (3.17), (3.18), (3.22) and (3.23) for \( k = 1, 2, \ldots, N - 1 \), or for \( k = 1, 2, \ldots, N \), according to the end point \( a_N \) being absorbing or reflecting, where \( a_{N+1} \) is read as \( t_2 = \infty \) in the latter case. In the same way as that in the proof of Proposition A.1, we can obtain the following. So we omit the proof.

Proposition B.1 Let \( i = 1, 2 \) and \( k = 2, \ldots, N \). Then \( \varphi_i(a_k, -\lambda) \) has \( k \) zeros \( \lambda_{i,j,k} \) such that

\[
0 < \lambda_{1,1,k} < \lambda_{2,1,k} < \lambda_{1,2,k} < \lambda_{2,2,k} < \cdots < \lambda_{1,k,k} < \lambda_{2,k,k}. \tag{B.1}
\]
Further
\[0 = \lambda_{1,1,0} < \lambda_{2,1,0}^+,\]
and it holds true that
\[0 = \lambda_{1,1,k} < \lambda_{1,2,k} < \lambda_{2,1,k}^+ < \lambda_{2,2,k} < \cdots < \lambda_{1,k+1,k} < \lambda_{2,k+1,k}^+;\]
\[0 = \lambda_{1,1,k} < \lambda_{1,2,k} < \lambda_{1,2,k}^+ < \cdots < \lambda_{1,k+1,k} < \lambda_{1,k,k+1};\]
\[0 = \lambda_{2,1,k} < \lambda_{2,2,k} < \lambda_{2,2,k}^+ < \cdots < \lambda_{2,k+1,k} < \lambda_{2,k,k+1};\]
for \(k = 1, 2, \ldots, N - 1\). In particular, if the end point \(a_N\) is reflecting, \(\varphi_i^{\text{ref}}(a_N, -\lambda), \ i = 1, 2,\) have \(N + 1\) zeros \(\lambda_{i,j,N}^+\), and (A.9), (A.10) and (A.11) hold for \(k = N\).

References

[10] T. Takemura, Elementary solutions of Bessel processes with boundary conditions, Ann. Reports of Graduate School of Humanities and Sciences, Nara Women’s University, 23 (2008), 265–278.
Transition probability densities of birth and
dead processes with finite state space

IIZUKA Masaru and TOMISAKI Matsuyo

We consider birth and death processes with finite state space consisting of \( N + 1 \) points \((N \geq 2)\). These processes can be treated as one-dimensional generalized diffusion processes and have transition probability densities with respect to a discrete measure called the speed measure. We give explicit representations of transition probability densities according to boundary conditions. Following a general theory on elementary solutions, we obtain integral representations of transition probability densities. Through these integral representations it is not easy to see, however, the effects of boundary conditions on transition probability densities. We give eigen function expansions of transition probability densities to see how boundary conditions affect transition probability densities.