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<td>タイトル</td>
<td>一方的なを持つ義理の側面におけるデータの伝播の形態の変化に関する研究</td>
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<tr>
<td>作者</td>
<td>嶋村 士子</td>
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State of boundaries for harmonic transforms of
one-dimensional generalized diffusion processes

TAKEMURA Tomoko *†

1 Introduction

Let \( m \) be a right continuous nondecreasing function on an open interval \( I = (l_1, l_2) \),
where \(-\infty \leq l_1 < l_2 \leq \infty\), \( s \) be a continuous increasing function on \( I \), and \( k \) be
a right continuous nondecreasing function on \( I \). We assume that the support of the
measure \( dm(x) \) on \( I \) induced by \( m(x) \) is equal to \( I \). For a function \( u \) on \( I \), we set
\( u(l_i) = \lim_{x_i \to l_i, x_i \in I} u(x) \) if there exists the limit, for \( i = 1, 2 \). We set \( I^* = I \cup \{x; x = l_i \text{ with } |m(l_i)| + |s(l_i)| + |k(l_i)| < \infty, \text{ } i = 1, 2\} \). Let us fix a point \( c_0 \in I \) arbitrarily and set

\[
J_{\mu, \nu}(x) = \int_{(c_0, x]} d\mu(y) \int_{(c_0, y]} d\nu(z),
\]

for \( x \in I \), where \( d\mu \) and \( d\nu \) are Borel measures on \( I \) and the integral \( \int_{(a,b]} \) is read as
\( -\int_{(b,a]} \) if \( a > b \). Following [1], we call the boundary \( l_i \) to be

- \((s, m, k)\)-regular if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- \((s, m, k)\)-exit if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) = \infty \),
- \((s, m, k)\)-entrance if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- \((s, m, k)\)-natural if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) = \infty \).

We note that

- if \( l_i \) is \((s, m, k)\)-regular, \(|(m + k)(l_i)| < \infty \) and \(|s(l_i)| < \infty \),
- if \( l_i \) is \((s, m, k)\)-exit, \(|(m + k)(l_i)| = \infty \) and \(|s(l_i)| < \infty \),
- if \( l_i \) is \((s, m, k)\)-entrance, \(|(m + k)(l_i)| < \infty \) and \(|s(l_i)| = \infty \),
- if \( l_i \) is \((s, m, k)\)-natural, \(|(m + k)(l_i)| = \infty \) or \(|s(l_i)| = \infty \).

Let \( D(G) \) be the space of all functions \( u \in L^2(I, m) \) which have continuous representatives
\( u \) (we use the same symbol) satisfying the following conditions:

i) There exist two constants \( A, B \) and a function \( h_u \in L^2(I, m) \) such that

\[
u(x) = A + Bs(x) + \int_{(c_0, x]} \{s(x) - s(y)\} h_u(y) dm(y) \]

\[
\quad + \int_{(c_0, x]} \{s(x) - s(y)\} u(y) dk(y), \quad x \in I.
\] (1.1)
ii) If $l_i$ is regular, then $u(l_i) = 0$ for each $i = 1, 2$.

By virtue of (1.1), $h_u$ is uniquely determined as a function of $L^2(I, m)$ if it exists. The operator $G$ from $D(G)$ into $L^2(I, m)$ is defined by $Gu = h_u$, and it is called the one-dimensional generalized diffusion operator with the speed measure $s$, the scale function $s$, and the killing measure $k$ (ODGDO with $(s, m, k)$ for short). In the following, for a measurable functions $u$ on $I$, $D_s u(x)$ stands for the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{(s, \varepsilon)} \{u(x + \varepsilon) - u(x)\} / \{s(x + \varepsilon) - s(x)\}$, provided it exists. It is obvious that $u \in D(G)$ has the right derivative $D_s u$ and it satisfies

$$D_s u(y) - D_s u(x) = \int_{\{x, y\}} Gu(z) dm(z) + \int_{\{x, y\}} u(z) dk(z), \quad x, y \in I.$$ 

So we sometimes use the symbol $G u = (dD_s u - u dk) / dm$. Following McKean [4] (see also Section 4.11 of [2]), we can define the fundamental solution $p(t, x, y)$ of the following equation.

$$\frac{\partial}{\partial t} p(t, x, y) = G p(t, x, y), \quad t > 0, \quad x, y \in I,$$

where $G$ is applied to $x$ or $y$.

It is known that $p(t, x, y)$ satisfies the following properties:

0 < $p(t, x, y) = p(t, y, x)$ is continuous on $I \times I \times (0, \infty)$,

$p(s + t, x, y) = \int_{l} p(s, x, z) p(t, z, y) dm(z), \quad s, t > 0, \quad x, y \in I$,

$p(t, l_i, y) = 0, \quad t > 0, \quad y \in I, \quad$ if $l_i$ is not entrance,

$D_s p(t, l_i, y) = 0, \quad t > 0, \quad y \in I, \quad$ if $l_i$ is entrance,

where $D_s p(t, x, y) = \lim_{\varepsilon \to 0} \{p(t, x + \varepsilon, y) - p(t, x, y)\} / \{s(x + \varepsilon) - s(x)\}$. It is also known that there exists a one-dimensional generalized diffusion process (ODGDP for brief) $D = [X(t) : t \geq 0, \; P_x : x \in I^*]$ such that

$$P_x(X(t) \in E) = \int_{E} p(t, x, y) dm(y), \quad t > 0, \quad x \in I^*, \quad E \in B(I^*).$$

By this reason, $p(t, x, y)$ is sometimes called the transition probability density with respect to $m$. The state of boundaries, that is, $(s, m, k)$ -regular, exit, entrance, and natural, suggest the behavior of the sample paths of $D$ having the ODGDO $G$ with $(s, m, k)$ as the generator (see [2]). For $\beta \geq 0$ let $H_{s, m, k, \beta}$ be the set of all positive functions $h_{\beta}$ satisfying

$$h_{\beta}(x) = h_{\beta}(c_0) + D_s h_{\beta}(c_0) \{s(x) - s(c_0)\} + \int_{\{c_0, x\}} \{s(x) - s(y)\} h_{\beta}(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.$$ 

We call $h_{\beta}$ a $\beta$ harmonic function for $G$. For $h \in H_{s, m, k, \beta}$, we set

$$s_h(x) = \int_{\{c_0, x\}} h(y)^{-1} ds(y), \quad (1.2)$$

$$m_h(x) = \int_{\{c_0, x\}} h(y)^2 dm(y), \quad (1.3)$$

$$p_h(t, x, y) = e^{-\beta t} p(t, x, y) / h(x) h(y).$$
Let $G_h$ be an ODGDO with $(s_h, m_h, 0)$, where 0 denotes the null measure. Let $D_h$ be an ODGDP with $G_h$ as the generator. Then $p_h(t, x, y)$ is the transition probability density of $D_h$ with respect to $m_h$. We call $D_h$ a harmonic transform of $D$. In this paper we study state of boundaries for $D_h$. Our main result is as follows.

**Theorem 1.1** Let $h \in \mathcal{H}_{s,m,k,\beta}$ and $i = 1, 2$.

(i) Suppose that $l_i$ is $(s, m, k)$ -regular or exit. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $l_i$ being $(s, m, k)$ -regular or exit.

(ii) Suppose that $l_i$ is $(s, m, k)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $m(h(l_i)) < \infty$ or $|m_h(l_i)| = \infty$.

(iii) Suppose that $l_i$ is $(s, m, k)$ -natural. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance or natural according to $J_{s_h,m_h}(l_i) < \infty$ or $J_{s_h,m_h}(l_i) = \infty$. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular, exit, or natural according to $|m(l_i)| < \infty$, $|m(l_i)| = \infty$ and $J_{s_h,m_h}(l_i) < \infty$, or $|m(l_i)| = \infty$ and $J_{s_h,m_h}(l_i) = \infty$.

The statements of the theorem are tabulated as follows.

<table>
<thead>
<tr>
<th>$h(l_i) = 0$</th>
<th>$h(l_i) \in (0, \infty)$</th>
<th>$h(l_i) = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(s, m, k)$ -regular</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -regular</td>
</tr>
<tr>
<td>Ex. 3.1</td>
<td>Ex. 3.2</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -exit</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -exit</td>
</tr>
<tr>
<td>Ex. 3.3</td>
<td>Ex. 3.4</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -entrance</td>
<td>$\emptyset$</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
</tr>
<tr>
<td>Ex. 3.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -natural</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
</tr>
<tr>
<td>if $J_{s_h,m_h}(l_i) &lt; \infty$</td>
<td>if $</td>
<td>m(l_i)</td>
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<tr>
<td>Ex. 3.6</td>
<td>Ex. 3.6</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -natural</td>
<td>$\emptyset$</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
</tr>
<tr>
<td>if $J_{s_h,m_h}(l_i) = \infty$</td>
<td>if $</td>
<td>m(l_i)</td>
</tr>
<tr>
<td>Ex. 3.4, Ex. 3.5</td>
<td>Ex. 3.6</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -natural</td>
<td>$(s_h, m_h, 0)$ -natural</td>
<td>$(s_h, m_h, 0)$ -natural</td>
</tr>
<tr>
<td>if $J_{s_h,m_h}(l_i) = \infty$</td>
<td>if $</td>
<td>m(l_i)</td>
</tr>
<tr>
<td>Ex. 3.5, Ex. 3.7</td>
<td></td>
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</tr>
</tbody>
</table>

The symbol $\emptyset$ of the table means that there don’t exist any $\beta$ harmonic functions for $G$ (see Lemma 2.1 below). We exhibit examples for each cases of the table in Section 3. Example 3.1 etc. are abbreviated as Ex.3.1 etc., respectively.

In [3] Maeno treated harmonic transforms different from ours. More precisely, let $s$ and $m$ be the scale function and speed measure on $I$. Let $\mathcal{M}_s^0$ be the set of all positive continuous functions $h$ on $I$ such that $h$ has the right derivative $D_s h$ which is right
continuous and nonincreasing. For \( h \in \mathcal{M}_\beta \) we consider \( s_h \) and \( m_h \) given by (1.2) and (1.3), respectively. Further set \( k_h(x) = - \int_{(c_n,x)} h dD_s h(x) \). Let \( \mathcal{G}_h^\beta \) be an ODGDO with \((s_h, m_h, k_h)\). She discusses the state of boundaries for \( N_h \) having the ODGDO \( \mathcal{G}_h^\beta \) as the generator. Since \( \mathcal{H}_{s,m,k,\beta} \cap \mathcal{M}_\beta = \emptyset \) if \( k \neq 0 \) or \( \beta > 0 \), we cannot derive Theorem 1.1 from her results. However there is a relation between our harmonic transform and Maeno's harmonic transform. We discuss this relation in [5].

2 Proof of main theorem

In this section we prove Theorem 1.1 for \( l_1 \). First we summarize some properties of \( \beta \) harmonic functions.

Lemma 2.1 ([2], [6]) Let \( h \in \mathcal{H}_{s,m,k,\beta} \).

(i) For \( l_1 < x < y < l_2 \),

\[
D_s h(x) \leq \frac{h(y) - h(x)}{s(y) - s(x)} \leq D_s h(y).
\]

(ii) Suppose \( l_1 \) is regular or exit. Then \( 0 \leq h(l_1) < \infty \). If \( h(l_1) = 0 \), \( h(x) \leq D_s h(x)(s(x) - s(l_1)) \).

(iii) Suppose \( l_1 \) is entrance. Then \( 0 < h(l_1) \leq \infty \). If \( h(l_1) = \infty \), then \( D_s h(l_1) \in [-\infty, 0) \), \( |s_h(l_1)| < \infty \), and \( \int_{l_1} h(y) dm(y) < \infty \).

(iv) Suppose \( l_1 \) is natural. Then \( h(l_1) = 0 \), or \( h(l_1) = \infty \). If \( h(l_1) = \infty \), \( |s_h(l_1)| < \infty \).

Remark 2.2 Suppose that \( h \in \mathcal{H}_{s,m,k,\beta} \) and \( 0 < h(l_1) < \infty \). Then

\[
0 < \lim_{x \rightarrow l_1} \frac{s_h(x)}{s(x)} < \infty, \quad 0 < \lim_{x \rightarrow l_1} \frac{m_h(x)}{m(x)} < \infty.
\]

The statements of Theorem 1.1 (i) and (ii) corresponding to \( 0 < h(l_1) < \infty \) are derived from Remark 2.2. We divide the proof of the theorem into three cases for expect \( 0 < h(l_1) < \infty \). In the following we fix \( s \), \( m \), \( k \), \( \beta \), and \( h \in \mathcal{H}_{s,m,k,\beta} \).

2.1 The case that \( l_1 \) is \((s, m, k)\) -regular or exit

Suppose that \( l_1 \) is \((s, m, k)\) -regular or exit. Then \( 0 \leq h(l_1) < \infty \). If \( h(l_1) = 0 \), by means of Lemma 2.1, there is an \( x_0 \in I \) such that

\[
h(x) \leq (s(x) - s(l_1)) D_s h(x_0), \quad l_1 < x < x_0.
\]

Therefore

\[
\int_{(l_1,x_0]} h^{-2}(x) ds(x) \geq (D_s h(x_0))^{-2} \int_{(l_1,x_0]} \frac{ds(x)}{(s(x) - s(l_1))^2} = \infty.
\]

Hence \( s_h(l_1) = -\infty \). Furthermore

\[
\int_{(l_1,x_0]} h^2(x) dm(x) \int_{(x,x_0]} h^{-2}(y) ds(y) \leq \int_{(l_1,x_0]} dm(x) \int_{(x,x_0]} ds(y) < \infty,
\]

which shows that \( l_1 \) is \((s_h, m_h, 0)\) -entrance.
2.2 The case that $l_1$ is $(s, m, k)$ -entrance

Suppose that $l_1$ is $(s, m, k)$ -entrance. If $h(x) = \infty$, then we have

$$\lim_{x \to l_1} \frac{\int_{(l_1, x]} h^{-2}(y) \, ds(y)}{h^{-1}(x)} = - \lim_{x \to l_1} \frac{1}{D_s h(x)} = - \frac{1}{D_s h(l_1)}.$$  

We note that $D_s h(l_1) \in [0, \infty)$ by means of Lemma 2.1. Hence there are $x_0 \in I$ and a positive constant $C$ such that $\int_{(l_1, x]} h^{-2}(y) \, ds(y) \leq C h^{-1}(x), l_1 < x < x_0$. Combining this with $|s_h(l_1)| < \infty$, we find

$$\int_{(l_1, x]} h^{-2}(y) \, ds(y) \int_{(x, x_0]} h^2(x) \, dm(x) = \int_{(l_1, x_0]} h^2(y) \, dm(y) \int_{(l_1, y]} h^{-2}(x) \, ds(x)$$
$$\leq C \int_{(l_1, c]} h(y) \, dm(y) < \infty.$$  

Thus we have that $l_1$ is $(s_h, m_h, 0)$ -regular (resp. -exit) if $|m_h(l_1)| < \infty$ (resp. $|m_h(l_1)| = \infty$).

2.3 The case that $l_1$ is $(s, m, k)$ -natural

Suppose that $l_1$ is $(s, m, k)$ -natural. Then $h(l_1) = 0$ or $h(l_1) = \infty$. Suppose that $h(l_1) = 0$. If $s(l_1) = -\infty$, for any $M > 0$ there exists an $x_0$ such that $h(x) > M$, $l_1 < x < x_0$. We have

$$\int_{(l_1, x_0]} h^{-2}(x) \, ds(x) \geq M^2 \int_{(l_1, x_0]} ds(x) = \infty.$$  

Hence we have $s_h(l_1) = -\infty$. If $s(l_1) > -\infty$, there exist an $x_1$ and a positive constant $C$ such that $h(x) \leq C(s(x) - s(l_1))$ for $l_1 < x < x_1$. Therefore

$$\int_{(l_1, x_1]} h^{-2}(x) \, ds(x) \geq \frac{1}{C} \int_{(l_1, x_1]} \frac{ds(x)}{(s(x) - s(l_1))^2} = \infty,$$  

Then we have $s_h(l_1) = -\infty$. Thus $l_1$ is $(s_h, m_h, 0)$ -entrance or natural according to $J_{m_h, s_h}(l_1) < \infty$ or $J_{m_h, s_h}(l_1) = \infty$.

If $h(l_1) = \infty$, then

$$\int_{(l_1, c]} h^{-2}(y) \, ds(y) < \infty,$$  

by means of Lemma 2.1. Therefore $l_1$ is $(s_h, m_h, 0)$ -regular, exit, or natural according to $|m_h(l_1)| < \infty$, $|m_h(l_1)| = \infty$ and $J_{s_h, m_h}(l_1) < \infty$, or $|m_h(l_1)| < \infty$ and $J_{s_h, m_h}(l_1) = \infty$.

3 Examples

In this section we give each examples in Table 1.
Example 3.1 Consider the generator
\[ G = x^2(x^2 - 1) \frac{d^2}{dx^2} + 2x^3 \frac{d}{dx} - x^2(x^2 - 1)^{-1}, \]
on \((1, \infty)\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = (x^2 - 1)^{-1} \, dx, \quad dm(x) = x^{-2} \, dx, \quad dk(x) = (x^2 - 1)^{-1} \, dx. \]
The end points 1 and \(\infty\) are \((s, m, k)\) -natural and \((s, m, k)\) -regular, respectively.

Set \(h(x) = (x^2 - 1)^{-\frac{3}{2}}\). \(h(x)\) is a 0 harmonic function for \(G\). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.
\[ ds_h(x) = dx, \quad dm_h(x) = x^{-2}(x^2 - 1)^{-1} \, dx, \quad G_h = x^2(x^2 - 1) \frac{d^2}{dx^2}. \]
Since \(J_{s_h, m_h}(1) = \infty\) and \(J_{m_h, s_h}(1) = \infty\), the end point 1 is \((s_h, m_h, 0)\) -natural. Since \(|s_h(\infty)| = \infty\) and \(J_{m_h, s_h}(\infty) < \infty\), the end point \(\infty\) is \((s_h, m_h, 0)\) -entrance.

Example 3.2 Consider the generator
\[ G = e^{\gamma x} \frac{d^2}{dx^2} - \kappa, \]
on \((0, \infty)\), where \(\gamma > 0\) and \(\kappa > 0\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = dx, \quad dm(x) = e^{-\gamma x} \, dx, \quad dk(x) = \kappa e^{-\gamma x} \, dx. \]
The end points 0 and \(\infty\) are \((s, m, k)\) -regular and \((s, m, k)\) -entrance, respectively.

For \(\lambda \geq 0\), set \(h(x) = K_0 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{x^2}{2}} \right)\). \(h(x)\) is a \(\lambda\) harmonic function for \(G\). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.
\[ ds_h(x) = K_0^{-2} \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{x^2}{2}} \right) \, dx, \quad dm_h(x) = K_0^{-2} \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{x^2}{2}} \right) e^{-\gamma x} \, dx, \]
\[ G_h = e^{\gamma x} \frac{d^2}{dx^2} + 2\sqrt{\lambda + \kappa} e^{\frac{x^2}{2}} \frac{d}{dx} K_0 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{x^2}{2}} \right) \]
\[ G_h = e^{\gamma x} \frac{d^2}{dx^2} + 2\sqrt{\lambda + \kappa} e^{\frac{x^2}{2}} \frac{d}{dx} K_0 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{x^2}{2}} \right). \]
Since \(|s_h(0)| < \infty\) and \(|m_h(0)| < \infty\), the end point 0 is \((s_h, m_h, 0)\) -regular. Since \(|s_h(\infty)| < \infty\) and \(|m_h(\infty)| < \infty\), the end point \(\infty\) is \((s_h, m_h, 0)\) -regular.

Example 3.3 Consider the generator
\[ G = x(x^2 - 1) \frac{d^2}{dx^2} + 2x^3 \frac{d}{dx} - x(x^2 - 1)^{-1}, \]
on \((1, \infty)\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = (x^2 - 1)^{-1} \, dx, \quad dm(x) = x^{-1} \, dx, \quad dk(x) = (x^2 - 1)^{-1} \, dx. \]
The end points 1 and \(\infty\) are \((s, m, k)\) -natural and \((s, m, k)\) -exit, respectively.
Set $h(x) = (x^2 - 1)^{-\frac{1}{2}}$. $h(x)$ is a 0 harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$d s_h(x) = dx, \quad d m_h(x) = x^{-1}(x^2 - 1)^{-1} dx, \quad G_h = x(x^2 - 1) \frac{d^2}{dx^2}. $$

Since $|m_h(1)| = \infty$ and $J_{s_h,m_h}(1) = \infty$, the end point 1 is $(s_h, m_h, 0)$ -natural. Since $|s_h(\infty)| = \infty$ and $J_{m_h,s_h}(\infty) < \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -entrance.

**Example 3.4** Consider the generator

$$G = x^\frac{3}{2} \frac{d^2}{dx^2} + \frac{2x^3}{(x + 1)(\log(x + 1) + 1)} \frac{d}{dx} - \frac{x^3}{(x + 1)^2(\log(x + 1) + 1)},$$

on $(0, \infty)$. The scale function, speed measure, and killing measure are given by

$$d s(x) = (\log(x + 1) + 1)^{-2} dx,$$

$$d m(x) = (\log(x + 1) + 1)^2 x^{-\frac{3}{2}} dx,$$

$$d k(x) = (\log(x + 1) + 1)(x + 1)^{-2} dx.$$

The end points 0 and $\infty$ are $(s, m, k)$ -exit and $(s, m, k)$ -natural, respectively.

Set $h(x) = (\log(x + 1) + 1)^{-4}$. $h(x)$ is a 0 harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$d s_h(x) = dx, \quad d m_h(x) = x^{-\frac{3}{2}} dx, \quad G_h = x^\frac{3}{2} \frac{d^2}{dx^2}. $$

Since $|m_h(0)| = \infty$ and $J_{s_h,m_h}(0) < \infty$, the end point 0 is $(s_h, m_h, 0)$ -exit. Since $|s_h(\infty)| = \infty$ and $J_{m_h,s_h}(\infty) = \infty$, the end point $\infty$ is $(s_h, m_h, 0)$ -natural.

**Example 3.5** Consider the generator

$$G = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \gamma,$$

on $(0, \infty)$, where $\gamma > 0$. The scale function, speed measure, and killing measure are given by

$$d s(x) = x^{-1} dx, \quad d m(x) = 2x dx, \quad d k(x) = 2\gamma x dx.$$

The end points 0 and $\infty$ are $(s, m, k)$ -entrance and $(s, m, k)$ -natural, respectively.

We consider two harmonic transforms for $G$.

For $\lambda \geq 0$, set $\phi(x) = x^{-1}\sinh(x\sqrt{2(\lambda + \gamma)})$. $\phi(x)$ is a $\lambda$ harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$d s_\phi(x) = x \sinh^{-2}(x\sqrt{2(\lambda + \gamma)}) dx, \quad d m_\phi(x) = 2x^{-1}\sinh^2(x\sqrt{2(\lambda + \gamma)}) dx,$$

$$G_\phi = \frac{1}{2} \frac{d^2}{dx^2} + \left( x^{-1} \sinh(x\sqrt{2(\lambda + \gamma)}) + \tanh^{-1}(x\sqrt{2(\lambda + \gamma)}) \right) \frac{d}{dx}.$$

Since $|s_\phi(0)| = \infty$ and $J_{m_\phi,s_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -entrance. Since $|m_\phi(\infty)| = \infty$ and $J_{s_\phi,m_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_\phi, m_\phi, 0)$ -natural.
For $A > 0$, set $\psi(x) = x^{-1/2} e^{x/A}$. $\psi(x)$ is a $A$ harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_\psi(x) = xe^{x/2} \sqrt{2(\lambda + \gamma)} dx, \quad dm_\psi(x) = x e^{-x/2} \sqrt{2(\lambda + \gamma)} dx, \quad G_\psi = \frac{1}{2} \frac{d^2}{dx^2} + \frac{d}{dx}.$$  

Since $|m_\psi(0)| = \infty$ and $J_{m_\psi, m_\psi}(0) < \infty$, the end point 0 is $(s_\psi, m_\psi, 0)$ -exit. Since $|s_\psi(\infty)| = \infty$ and $J_{m_\psi, s_\psi}(\infty) = \infty$, the end point $\infty$ is $(s_\psi, m_\psi, 0)$ -natural.

Example 3.6 Consider the generator

$$G = \frac{1}{2} \frac{d^2}{dx^2} - \frac{\gamma^2 - 2^{-2}}{2x^2},$$

on $(0, \infty)$, where $|\gamma| > \frac{1}{2}$. The scale function, speed measure, and killing measure are given by

$$ds(x) = dx, \quad dm(x) = 2 dx, \quad dk(x) = (\gamma^2 - 2^{-2}) x^{-2} dx.$$  

The end points 0 and $\infty$ are $(s, m, k)$ -natural.

We consider two harmonic transforms for $G$.

For $\lambda > 0$, set $\phi(x) = \sqrt{x} I_\gamma(x \sqrt{2\lambda})$. $\phi(x)$ is a $A$ harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_\phi(x) = x^{-1} I_\gamma^{-1}(x \sqrt{2\lambda}) dx, \quad dm_\phi(x) = 2 x I_\gamma^2(x \sqrt{2\lambda}) dx,$$

$$G_\phi = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1}{2x} + \frac{\gamma}{x \sqrt{2\lambda}} + \frac{I_{\gamma+1}(x \sqrt{2\lambda})}{I_\gamma(x \sqrt{2\lambda})} \right) \frac{d}{dx}.$$  

Since $|s_\phi(0)| = \infty$ and $J_{m_\phi, s_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -entrance. Since $|s_\phi(\infty)| = \infty$ and $J_{m_\phi, s_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_\phi, m_\phi, 0)$ -natural.

For $\lambda > 0$ set $\psi(x) = \sqrt{x} K_\gamma(x \sqrt{2\lambda})$. $\psi(x)$ is a $A$ harmonic function for $G$. We obtain the transformed scale function and the transformed speed measure as follows.

$$ds_\psi(x) = x^{-1} K_\gamma^{-1}(x \sqrt{2\lambda}) dx, \quad dm_\psi(x) = 2 x K_\gamma^2(x \sqrt{2\lambda}) dx,$$

$$G_\psi = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1}{2x} + \frac{\gamma}{x \sqrt{2\lambda}} - \frac{K_{\gamma+1}(x \sqrt{2\lambda})}{K_\gamma(x \sqrt{2\lambda})} \right) \frac{d}{dx}.$$  

Since $|m_\psi(0)| = \infty$ and $J_{m_\psi, m_\psi}(0) < \infty$, the end point 0 is $(s_\psi, m_\psi, 0)$ -exit. Since $|s_\psi(\infty)| = \infty$ and $J_{m_\psi, s_\psi}(\infty) = \infty$, the end point $\infty$ is $(s_\psi, m_\psi, 0)$ -natural.

Example 3.7 Consider the generator

$$G = \frac{1}{2} \frac{d^2}{dx^2} + \frac{x^3}{2} \frac{d}{dx} - \frac{1}{8} x^2,$$

on $(0, \infty)$. The scale function, speed measure, and killing measure are given by

$$ds(x) = x^{-1} dx, \quad dm(x) = 2 x^{-3} dx, \quad dk(x) = \frac{1}{4} x^{-1} dx.$$  

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The end points 0 and \( \infty \) are \((s, m, k)\)-natural.

Set \( h(x) = x^\frac{1}{2} \). \( h(x) \) is a 0 harmonic function for \( \mathcal{G} \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

\[
d_{sh}(x) = x^{-2} \, dx, \quad dm_{h}(x) = 2x^{-2} \, dx, \quad G_{h} = \frac{x^4}{2} \frac{d^2}{dx^2} + x^3 \frac{d}{dx}.
\]

Since \( |s_h(0)| = \infty \) and \( |m_h(0)| = \infty \), the end point 0 is \((s_h, m_h, 0)\)-natural. Since \( |s_h(\infty)| < \infty \) and \( |m_h(\infty)| < \infty \), the end point \( \infty \) is \((s_h, m_h, 0)\)-regular.

References


State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes

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We consider a one-dimensional generalized diffusion operator $\mathcal{G}$ represented by triplet of Borel measures and a harmonic transform $\mathcal{G}_h$ of $\mathcal{G}$, where $h$ is a harmonic function for $\mathcal{G}$. Specially we treat an operator with killing measure which is not null measure. We consider the state of boundaries for the one-dimensional generalized diffusion process $\mathcal{D}_h$ with $\mathcal{G}_h$ as the generator. State of boundaries for $\mathcal{D}_h$ may be different from those for $\mathcal{D}$ which is a one-dimensional generalized process with $\mathcal{G}$ as the generator. We characterize the state of boundaries for $\mathcal{D}_h$ in terms of the Borel measures and a harmonic function for $\mathcal{G}$. After we prove our main theorem, we give some examples.