<table>
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<th>内容</th>
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<td>テーマ</td>
<td>一次元一般拡散過程のハーモニック変換の境界の影響に関する研究</td>
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<tr>
<td>作者</td>
<td>嶽村 智子</td>
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State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes

TAKEMURA Tomoko *

1 Introduction

Let $m$ be a right continuous nondecreasing function on an open interval $I = (l_1, l_2)$, where $-\infty < l_1 < l_2 < \infty$, $s$ be a continuous increasing function on $I$, and $k$ be a right continuous nondecreasing function on $I$. We assume that the support of the measure $dm(x)$ on $I$ induced by $m(x)$ is equal to $I$. For a function $u$ on $I$, we set $u(l_i) = \lim_{x \to l_i, x \in I} u(x)$ if there exists the limit, for $i = 1, 2$. We set $I^* = I \cup \{x; x = l_i$ with $|m(l_i)| + |s(l_i)| + |k(l_i)| < \infty, i = 1, 2\}$. Let us fix a point $c_0 \in I$ arbitrarily and set

$$J_{\mu, \nu}(x) = \int_{[c_0, x]} d\mu(y) \int_{[c_0, y]} d\nu(z),$$

for $x \in I$, where $d\mu$ and $d\nu$ are Borel measures on $I$, and the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [1], we call the boundary $l_i$ to be

- $(s, m, k)$-regular if $J_{s, m+k}(l_i) < \infty$ and $J_{m+k, s}(l_i) < \infty$,
- $(s, m, k)$-exit if $J_{s, m+k}(l_i) < \infty$ and $J_{m+k, s}(l_i) = \infty$,
- $(s, m, k)$-entrance if $J_{s, m+k}(l_i) = \infty$ and $J_{m+k, s}(l_i) < \infty$,
- $(s, m, k)$-natural if $J_{s, m+k}(l_i) = \infty$ and $J_{m+k, s}(l_i) = \infty$.

We note that

- if $l_i$ is $(s, m, k)$-regular, $|(m+k)(l_i)| < \infty$ and $|s(l_i)| < \infty$,
- if $l_i$ is $(s, m, k)$-exit, $|(m+k)(l_i)| = \infty$ and $|s(l_i)| < \infty$,
- if $l_i$ is $(s, m, k)$-entrance, $|(m+k)(l_i)| < \infty$ and $|s(l_i)| = \infty$,
- if $l_i$ is $(s, m, k)$-natural, $|(m+k)(l_i)| = \infty$ or $|s(l_i)| = \infty$.

Let $D(\mathcal{G})$ be the space of all functions $u \in L^2(I, m)$ which have continuous representatives $u$ (we use the same symbol) satisfying the following conditions:

i) There exist two constants $A$, $B$ and a function $h_u \in L^2(I, m)$ such that

$$u(x) = A + Bs(x) + \int_{(c_0, x]} \{s(x) - s(y)\} h_u(y) dm(y)$$

$$+ \int_{(c_0, x]} \{s(x) - s(y)\} u(y) dk(y), \quad x \in I. \quad (1.1)$$

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ii) If \( l_i \) is regular, then \( u(l_i) = 0 \) for each \( i = 1, 2 \).

By virtue of (1.1), \( h_u \) is uniquely determined as a function of \( L^2(I, m) \) if it exists. The operator \( G \) from \( D(G) \) into \( L^2(I, m) \) is defined by \( Gu = h_u \), and it is called the one-dimensional generalized diffusion operator with the speed measure \( m \), the scale function \( s \), and the killing measure \( k \) (ODGDO with \( (s, m, k) \) for short). In the following, for a measurable functions \( u \) on \( I \), \( D_s u(x) \) stands for the right derivative with respect to \( s(x) \), that is, \( D_s u(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon) - u(x)}{s(x + \varepsilon) - s(x)} \), provided it exists. It is obvious that \( u \in D(G) \) has the right derivative \( D_s u \) and it satisfies

\[
D_s u(y) - D_s u(x) = \int_{\{x,y\}} G u(z) \, dm(z) + \int_{\{x,y\}} u(z) \, dk(z), \quad x, y \in I.
\]

So we sometimes use the symbol \( Gu = (dD_s u - u dk)/dm \). Following McKean [4] (see also Section 4.11 of [2]), we can define the fundamental solution \( p(t, x, y) \) of the following equation.

\[
\frac{\partial}{\partial t} p(t, x, y) = G p(t, x, y), \quad t > 0, \quad x, y \in I,
\]

where \( G \) is applied to \( x \) or \( y \).

It is known that \( p(t, x, y) \) satisfies the following properties:

\[
0 < p(t, x, y) = p(t, y, x) \text{ is continuous on } I \times I \times (0, \infty),
\]

\[
p(s + t, x, y) = \int_I p(s, z, y) p(t, y, z) \, dm(z), \quad s, t > 0, \quad x, y \in I,
\]

\[
p(t, l_i, y) = 0, \quad t > 0, \quad y \in I, \quad \text{if } l_i \text{ is not entrance},
\]

\[
D_s p(t, l_i, y) = 0, \quad t > 0, \quad y \in I, \quad \text{if } l_i \text{ is entrance},
\]

where \( D_s p(t, x, y) = \lim_{\varepsilon \to 0} \{p(t, x + \varepsilon, y) - p(t, x, y)\} / \{s(x + \varepsilon) - s(x)\} \). It is also known that there exists a one-dimensional generalized diffusion process (ODGDP for brief) \( I = \{X(t) : t \geq 0, \quad P_x : x \in I^*\} \) such that

\[
P_x(X(t) \in E) = \int_E p(t, x, y) \, dm(y), \quad t > 0, \quad x \in I^*, \quad E \in \mathcal{B}(I^*).
\]

By this reason, \( p(t, x, y) \) is sometimes called the transition probability density with respect to \( m \). The state of boundaries, that is, \( (s, m, k) \)-regular, exit, entrance, and natural, suggest the behavior of the sample paths of \( I \) having the ODGDO \( G \) with \( (s, m, k) \) as the generator (see [2]). For \( \beta \geq 0 \) let \( H_{s,m,k,\beta} \) be the set of all positive functions \( h_\beta \) satisfying

\[
h_\beta(x) = h_\beta(c_o) + D_s h_\beta(c_o) \{s(x) - s(c_o)\}
+ \int_{\{c_o,x\}} \{s(x) - s(y)\} h_\beta(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.
\]

We call \( h_\beta \) a \( \beta \) harmonic function for \( G \). For \( h \in H_{s,m,k,\beta} \), we set

\[
s_h(x) = \int_{\{c_o,x\}} h(y)^{-2} ds(y), \quad (1.2)
\]

\[
m_h(x) = \int_{\{c_o,x\}} h(y)^2 dm(y), \quad (1.3)
\]

\[
p_h(t, x, y) = e^{-\beta t} p(t, x, y) / h(x) h(y).
\]
Let $G_h$ be an ODGDO with $(s_h, m_h, 0)$, where 0 denotes the null measure. Let $D_h$ be an ODGDP with $G_h$ as the generator. Then $p_h(t, x, y)$ is the transition probability density of $D_h$ with respect to $m_h$. We call $D_h$ a harmonic transform of $D$. In this paper we study state of boundaries for $D_h$. Our main result is as follows.

**Theorem 1.1** Let $h \in H_{s, m, k, \beta}$ and $i = 1, 2$.

(i) Suppose that $l_i$ is $(s, m, k)$ -regular or exit. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $l_i$ being $(s, m, k)$ -regular or exit.

(ii) Suppose that $l_i$ is $(s, m, k)$ -entrance. If $0 < h(l_i) < \infty$, then $l_i$ is $(s_h, m_h, 0)$ -entrance. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular or exit according to $|m_h(l_i)| < \infty$ or $|m_h(l_i)| = \infty$.

(iii) Suppose that $l_i$ is $(s, m, k)$ -natural. If $h(l_i) = 0$, then $l_i$ is $(s_h, m_h, 0)$ -entrance or natural according to $J_{m_h, s_h}(l_i) < \infty$ or $J_{m_h, s_h}(l_i) = \infty$. If $h(l_i) = \infty$, then $l_i$ is $(s_h, m_h, 0)$ -regular, exit, or natural according to $|m(l_i)| < \infty$, $|m(l_i)| = \infty$ and $J_{s_h, m_h}(l_i) < \infty$, or $|m(l_i)| = \infty$ and $J_{s_h, m_h}(l_i) = \infty$.

The statements of the theorem are tabulated as follows.

<table>
<thead>
<tr>
<th></th>
<th>$h(l_i) = 0$</th>
<th>$h(l_i) \in (0, \infty)$</th>
<th>$h(l_i) = \infty$</th>
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<tbody>
<tr>
<td>$(s, m, k)$ -regular</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -regular</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td>Ex. 3.1</td>
<td>Ex. 3.2</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -exit</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -exit</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td>Ex. 3.3</td>
<td>Ex. 3.4</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -entrance</td>
<td>$\emptyset$</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex. 3.5</td>
<td>Ex. 3.5</td>
<td></td>
</tr>
<tr>
<td>$(s, m, k)$ -natural</td>
<td>$(s_h, m_h, 0)$ -entrance</td>
<td>$(s_h, m_h, 0)$ -regular</td>
<td></td>
</tr>
<tr>
<td></td>
<td>if $</td>
<td>m_h(l_i)</td>
<td>&lt; \infty$</td>
</tr>
<tr>
<td></td>
<td>Ex. 3.7</td>
<td>Ex. 3.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(s_h, m_h, 0)$ -exit</td>
<td>$(s_h, m_h, 0)$ -exit</td>
<td></td>
</tr>
<tr>
<td></td>
<td>if $</td>
<td>m_h(l_i)</td>
<td>= \infty$</td>
</tr>
<tr>
<td></td>
<td>Ex. 3.5</td>
<td>Ex. 3.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(s_h, m_h, 0)$ -natural</td>
<td>$(s_h, m_h, 0)$ -natural</td>
<td></td>
</tr>
<tr>
<td></td>
<td>if $J_{m_h, s_h}(l_i) &lt; \infty$</td>
<td>if $J_{m_h, s_h}(l_i) &lt; \infty$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex. 3.6</td>
<td>Ex. 3.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>if $J_{m_h, s_h}(l_i) = \infty$</td>
<td>if $</td>
<td>m(l_i)</td>
</tr>
<tr>
<td></td>
<td>Ex. 3.4, Ex. 3.5</td>
<td>Ex. 3.5, Ex. 3.6</td>
<td></td>
</tr>
</tbody>
</table>

The symbol $\emptyset$ of the table means that there don't exist any $\beta$ harmonic functions for $G$ (see Lemma 2.1 below). We exhibit examples for each cases of the table in Section 3. Example 3.1 etc. are abbreviated as Ex.3.1 etc., respectively.

In [3] Maeno treated harmonic transforms different from ours. More precisely, let $s$ and $m$ be the scale function and speed measure on $I$. Let $M^*_\beta$ be the set of all positive continuous functions $h$ on $I$ such that $h$ has the right derivative $D_s h$ which is right
continuous and nonincreasing. For \( h \in \mathcal{M}_\alpha \) we consider \( s_h \) and \( m_h \) given by (1.2) and (1.3), respectively. Further set \( k_h(x) = -\int_{(s_h, x]} \delta h \, ds_h(x) \). Let \( \mathcal{G}_h \) be an ODGDO with \((s_h, m_h, k_h)\). She discusses the state of boundaries for \( N \) having the ODGDO \( \mathcal{G}_h \) as the generator. Since \( \mathcal{H}_{s,m,k,\beta} \cap \mathcal{M}_\alpha = \emptyset \) if \( k \neq 0 \) or \( \beta > 0 \), we cannot derive Theorem 1.1 from her results. However there is a relation between our harmonic transform and Maeno’s harmonic transform. We discuss this relation in [5].

2 Proof of main theorem

In this section we prove Theorem 1.1 for \( l_1 \). First we summarize some properties of \( \beta \) harmonic functions.

Lemma 2.1 ([2], [6]) Let \( h \in \mathcal{H}_{s,m,k,\beta} \).

(i) For \( l_1 < x < y < l_2 \),

\[
D_s h(x) \leq \frac{h(y) - h(x)}{s(y) - s(x)} \leq D_s h(y).
\]

(ii) Suppose \( l_1 \) is regular or exit. Then \( 0 \leq h(l_1) < \infty \). If \( h(l_1) = 0 \), \( h(x) \leq D_s h(x)(s(x) - s(l_1)) \).

(iii) Suppose \( l_1 \) is entrance. Then \( 0 < h(l_1) \leq \infty \). If \( h(l_1) = \infty \), then \( D_s h(l_1) \in [-\infty, 0) \), \( |s_h(l_1)| \leq \infty \), and \( \int_{(l_1, c_0)} h(y) \, dm(y) < \infty \).

(iv) Suppose \( l_1 \) is natural. Then \( h(l_1) = 0 \), or \( h(l_1) = \infty \). If \( h(l_1) = \infty \), \( |s_h(l_1)| < \infty \).

Remark 2.2 Suppose that \( h \in \mathcal{H}_{s,m,k,\beta} \) and \( 0 < h(l_1) < \infty \). Then

\[
0 < \lim_{x \downarrow l_1} \frac{s_h(x)}{s(x)} < \infty, \quad 0 < \lim_{x \downarrow l_1} \frac{m_h(x)}{m(x)} < \infty.
\]

The statements of Theorem 1.1 (i) and (ii) corresponding to \( 0 < h(l_1) < \infty \) are derived from Remark 2.2. We divide the proof of the theorem into three cases for expect \( 0 < h(l_1) < \infty \). In the following we fix \( s, m, k, \beta, \) and \( h \in \mathcal{H}_{s,m,k,\beta} \).

2.1 The case that \( l_1 \) is \((s, m, k)\) -regular or exit

Suppose that \( l_1 \) is \((s, m, k)\) -regular or exit. Then \( 0 \leq h(l_1) < \infty \). If \( h(l_1) = 0 \), by means of Lemma 2.1, there is an \( x_0 \in I \) such that

\[
h(x) \leq (s(x) - s(l_1))D_s h(x_0), \quad l_1 < x < x_0.
\]

Therefore

\[
\int_{(l_1, x_0]} h^{-2}(x) \, ds(x) \geq (D_s h(x_0))^{-2} \int_{(l_1, x_0]} \frac{ds(x)}{(s(x) - s(l_1))^2} = \infty.
\]

Hence \( s_h(l_1) = -\infty \). Furthermore

\[
\int_{(l_1, x_0]} h^2(x) \, dm(x) \int_{(x_0, l_1]} h^{-2}(y) \, ds(y) \leq \int_{(l_1, x_0]} dm(x) \int_{(x_0, l_1]} ds(y) < \infty,
\]

which shows that \( l_1 \) is \((s_h, m_h, 0)\) -entrance.
2.2 The case that \( l_1 \) is \((s, m, k)\) -entrance

Suppose that \( l_1 \) is \((s, m, k)\) -entrance. If \( h(x) = \infty \), then we have

\[
\lim_{x \to l_1} \frac{\int_{(l_1, x]} h^{-2}(y) \, ds(y)}{h^{-1}(x)} = \lim_{x \to l_1} \frac{1}{D_s h(x)} = \frac{1}{D_s h(l_1)}.
\]

We note that \( D_s h(l_1) \in [-\infty, 0) \) by means of Lemma 2.1. Hence there are \( x_0 \in I \) and a positive constant \( C \) such that \( \int_{(l_1, x]} h^{-2}(y) \, ds(y) \leq Ch^{-1}(x), \ l_1 < x < x_0 \). Combing this with \( |s_h(l_1)| < \infty \), we find

\[
\int_{(l_1, x_0]} h^{-2}(y) \, ds(y) \int_{(y, x_0]} h^2(x) \, dm(x) = \int_{(l_1, x_0]} h^2(y) \, dm(y) \int_{(l_1, y]} h^{-2}(x) \, ds(x) \\
\leq C \int_{(l_1, c]} h(y) \, dm(y) < \infty.
\]

Thus we have that \( l_1 \) is \((s_h, m_h, 0)\) -regular (resp. -exit) if \( |m_h(l_1)| < \infty \) (resp. \( |m_h(l_1)| = \infty \)).

2.3 The case that \( l_1 \) is \((s, m, k)\) -natural

Suppose that \( l_1 \) is \((s, m, k)\) -natural. Then \( h(l_1) = 0 \) or \( h(l_1) = \infty \).

Suppose that \( h(l_1) = 0 \). If \( s(l_1) = -\infty \), for any \( M > 0 \) there exists an \( x_0 \) such that \( h^{-1}(x) > M, \ l_1 < x < x_0 \). We have

\[
\int_{(l_1, x_0]} h^{-2}(y) \, ds(y) \geq M^2 \int_{(l_1, x_0]} ds(x) = \infty.
\]

Hence we have \( s_h(l_1) = -\infty \). If \( s(l_1) > -\infty \), there exist an \( x_1 \) and a positive constant \( C \) such that \( h(x) \leq C(s(x) - s(l_1)) \) for \( l_1 < x < x_1 \). Therefore

\[
\int_{(l_1, x_1]} h^{-2}(x) \, ds(x) \geq \frac{1}{C} \int_{(l_1, x_1]} \frac{ds(x)}{(s(x) - s(l_1))^2} = \infty,
\]

Then we have \( s_h(l_1) = -\infty \). Thus \( l_1 \) is \((s_h, m_h, 0)\) -entrance or natural according to \( J_{m_h,s_h}(l_1) < \infty \) or \( J_{m_h,s_h}(l_1) = \infty \).

If \( h(l_1) = \infty \), then

\[
\int_{(l_1, c]} h^{-2}(y) \, ds(y) < \infty,
\]

by means of Lemma 2.1. Therefore \( l_1 \) is \((s_h, m_h, 0)\) -regular, exit, or natural according to \( |m_h(l_1)| < \infty, \ |m_h(l_1)| = \infty \) and \( J_{s_h,m_h}(l_1) < \infty, \ |m_h(l_1)| < \infty \) and \( J_{s_h,m_h}(l_1) = \infty \).

3 Examples

In this section we give each examples in Table 1.
Example 3.1 Consider the generator
\[ G = x^2(x^2 - 1) \frac{d^2}{dx^2} + 2x^3 \frac{d}{dx} - x(x^2 - 1)^{-1}, \]
on \((1, \infty)\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = (x^2 - 1)^{-1} dx, \quad dm(x) = x^{-2} dx, \quad dk(x) = (x^2 - 1)^{-1} dx. \]
The end points 1 and \(\infty\) are \((s, m, k)\)-natural and \((s, m, k)\)-regular, respectively.

Set \( h(x) = (x^2 - 1)^{-\frac{1}{2}} \). \( h(x) \) is a 0 harmonic function for \( G \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.
\[ ds_h(x) = dx, \quad dm_h(x) = x^{-2}(x^2 - 1)^{-1} dx, \quad G_h = x^2(x^2 - 1) \frac{d^2}{dx^2}. \]
Since \( J_{s_h, m_h}(1) = \infty \) and \( J_{m_h, s_h}(1) = \infty \), the end point 1 is \((s_h, m_h, 0)\)-natural. Since \( |s_h(\infty)| = \infty \) and \( J_{m_h, s_h}(\infty) < \infty \), the end point \(\infty\) is \((s_h, m_h, 0)\)-entrance.

Example 3.2 Consider the generator
\[ G = e^{\gamma x} \frac{d^2}{dx^2} - \kappa, \]
on \((0, \infty)\), where \(\gamma > 0\) and \(\kappa > 0\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = dx, \quad dm(x) = e^{-\gamma x} dx, \quad dk(x) = \kappa e^{-\gamma x} dx. \]
The end points 0 and \(\infty\) are \((s, m, k)\)-regular and \((s, m, k)\)-entrance, respectively.

For \(\lambda \geq 0\), set \( h(x) = K_0 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{\gamma x}{2}} \right) \). \( h(x) \) is a \(\lambda\) harmonic function for \( G \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.
\[ ds_h(x) = K_0^{-2} \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{\gamma x}{2}} \right) dx, \quad dm_h(x) = K_0^2 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{\gamma x}{2}} \right) e^{-\gamma x} dx, \]
\[ G_h = e^{\gamma x} \frac{d^2}{dx^2} + 2\sqrt{\lambda + \kappa} e^{\frac{\gamma x}{2}} K_0 \left( \frac{2\sqrt{\lambda + \kappa}}{\gamma} e^{-\frac{\gamma x}{2}} \right) \frac{d}{dx}. \]
Since \(|s_h(0)| < \infty\) and \(|m_h(0)| < \infty\), the end point 0 is \((s_h, m_h, 0)\)-regular. Since \(|s_h(\infty)| < \infty\) and \(|m_h(\infty)| < \infty\), the end point \(\infty\) is \((s_h, m_h, 0)\)-regular.

Example 3.3 Consider the generator
\[ G = x(x^2 - 1) \frac{d^2}{dx^2} + 2x^3 \frac{d}{dx} - x(x^2 - 1)^{-1}, \]
on \((1, \infty)\). The scale function, speed measure, and killing measure are given by
\[ ds(x) = (x^2 - 1)^{-1} dx, \quad dm(x) = x^{-1} dx, \quad dk(x) = (x^2 - 1)^{-1} dx. \]
The end points 1 and \(\infty\) are \((s, m, k)\)-natural and \((s, m, k)\)-exit, respectively.
Set \( h(x) = (x^2 - 1)^{-\frac{1}{2}} \). \( h(x) \) is a 0 harmonic function for \( \mathcal{G} \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

\[
ds_h(x) = dx, \quad dm_h(x) = x^{-1}(x^2 - 1)^{-1} \, dx, \quad \mathcal{G}_h = x(x^2 - 1) - \frac{d^2}{dx^2}.
\]

Since \( |m_h(1)| = \infty \) and \( J_{s_h,m_h}(1) = \infty \), the end point 1 is \((s_h, m_h, 0)\)-natural. Since \( |s_h(\infty)| = \infty \) and \( J_{m_h,s_h}(\infty) < \infty \), the end point \( \infty \) is \((s_h, m_h, 0)\)-entrance.

**Example 3.4** Consider the generator

\[
\mathcal{G} = x^{\frac{3}{2}} \frac{d^2}{dx^2} + \frac{2x^{\frac{3}{2}}}{(x + 1)(\log(x + 1) + 1)} \frac{d}{dx} - \frac{x^{\frac{3}{2}}}{(x + 1)^2(\log(x + 1) + 1)},
\]
on \((0, \infty)\). The scale function, speed measure, and killing measure are given by

\[
ds_s(x) = (\log(x + 1) + 1)^{-2} \, dx,
\]
\[
dm(x) = (\log(x + 1) + 1)^2 x^{-2} \, dx,
\]
\[
dk(x) = (\log(x + 1) + 1)(x + 1)^{-2} \, dx.
\]
The end points 0 and \( \infty \) are \((s, m, k)\)-exit and \((s, m, k)\)-natural, respectively.

Set \( h(x) = (\log(x + 1) + 1)^{-4} \). \( h(x) \) is a 0 harmonic function for \( \mathcal{G} \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

\[
ds_h(x) = dx, \quad dm_h(x) = x^{-2} \, dx, \quad \mathcal{G}_h = x^{\frac{3}{2}} \frac{d^2}{dx^2}.
\]

Since \( |m_h(0)| = \infty \) and \( J_{s_h,m_h}(0) < \infty \), the end point 0 is \((s_h, m_h, 0)\)-exit. Since \( |s_h(\infty)| = \infty \) and \( J_{m_h,s_h}(\infty) = \infty \), the end point \( \infty \) is \((s_h, m_h, 0)\)-natural.

**Example 3.5** Consider the generator

\[
\mathcal{G} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \gamma,
\]
on \((0, \infty)\), where \( \gamma > 0 \). The scale function, speed measure, and killing measure are given by

\[
ds_s(x) = x^{-1} \, dx, \quad dm(x) = 2x \, dx, \quad dk(x) = 2\gamma x \, dx.
\]
The end points 0 and \( \infty \) are \((s, m, k)\)-entrance and \((s, m, k)\)-natural, respectively.

We consider two harmonic transforms for \( \mathcal{G} \).

For \( \lambda \geq 0 \), set \( \phi(x) = x^{-1} \sinh(x\sqrt{2(\lambda + \gamma)}) \). \( \phi(x) \) is a \( \lambda \) harmonic function for \( \mathcal{G} \). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

\[
ds_{\phi}(x) = x \sinh^{-2}(x\sqrt{2(\lambda + \gamma)}) \, dx, \quad dm_{\phi}(x) = 2x^{-1} \sinh^2(x\sqrt{2(\lambda + \gamma)}) \, dx, \quad \mathcal{G}_{\phi} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1}{x} - \frac{1}{x} \sinh(x\sqrt{2(\lambda + \gamma)}) + \tanh^{-1}(x\sqrt{2(\lambda + \gamma)}) \right) \frac{d}{dx}.
\]

Since \( |s_{\phi}(0)| = \infty \) and \( J_{m_{\phi},s_{\phi}}(0) < \infty \), the end point 0 is \((s_{\phi}, m_{\phi}, 0)\)-entrance. Since \( |m_{\phi}(\infty)| = \infty \) and \( J_{s_{\phi},m_{\phi}}(\infty) = \infty \), the end point \( \infty \) is \((s_{\phi}, m_{\phi}, 0)\)-natural.
For $\lambda > 0$, set $\psi(x) = x^{-1}e^{-x^{2/\lambda}}$. $\psi(x)$ is a $\lambda$ harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_\psi(x) = xe^{2x\sqrt{2/\lambda}}dx, \quad dm_\psi(x) = 2xe^{-2x\sqrt{2/\lambda}}dx, \quad G_\psi = \frac{1}{2\lambda} \frac{d^2}{dx^2} + \frac{d}{dx}.$$ 

Since $|m_\psi(0)| = \infty$ and $J_{s_\psi,m_\psi}(0) < \infty$, the end point 0 is $(s_\psi, m_\psi, 0)$ -exit. Since $|s_\psi(\infty)| = \infty$ and $J_{m_\psi,m_\psi}(\infty) = \infty$, the end point $\infty$ is $(s_\psi, m_\psi, 0)$ -natural.

**Example 3.6** Consider the generator

$$G = \frac{1}{2\lambda} \frac{d^2}{dx^2} - \frac{\gamma^2 - 2^{-2}}{2\lambda^2},$$

on $(0, \infty)$, where $|\gamma| > \frac{1}{2}$. The scale function, speed measure, and killing measure are given by

$$ds(x) = dx, \quad dm(x) = 2dx, \quad dk(x) = (\gamma^2 - 2^{-2})x^{-2}dx.$$ 

The end points 0 and $\infty$ are $(s, m, k)$ -natural.

We consider two harmonic transforms for $G$.

For $\lambda > 0$, set $\phi(x) = \sqrt{2/\lambda} I_{s_\lambda}(x\sqrt{2})$. $\phi(x)$ is a $\lambda$ harmonic function for $G$. We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

$$ds_\phi(x) = x^{-1}I_{s_\lambda}^{-2}(x\sqrt{2})dx, \quad dm_\phi(x) = 2xI_{s_\lambda}^2(x\sqrt{2})dx,$$

$$G_\phi = \frac{1}{2\lambda} \frac{d^2}{dx^2} + \left( \frac{1}{2} \frac{\gamma - 2^{-2}}{2\lambda^2} + I_{s_\lambda+1}(x\sqrt{2}) \right) \frac{d}{dx}.$$

Since $|s_\phi(0)| = \infty$ and $J_{m_\phi,s_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -entrance. Since $|m_\phi(\infty)| = \infty$ and $J_{s_\phi,m_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_\phi, m_\phi, 0)$ -natural.

For $\lambda > 0$ set $\psi(x) = \sqrt{2} K_{s_\lambda}(x\sqrt{2})$. $\psi(x)$ is a $\lambda$ harmonic function for $G$. We obtain the transformed scale function and the transformed speed measure as follows.

$$ds_\phi(x) = x^{-1}K_{s_\lambda}^{-2}(x\sqrt{2})dx, \quad dm_\phi(x) = 2xK_{s_\lambda}^2(x\sqrt{2})dx,$$

$$G_\phi = \frac{1}{2\lambda} \frac{d^2}{dx^2} + \left( \frac{1}{2} \frac{\gamma - 2^{-2}}{2\lambda^2} + K_{s_\lambda+1}(x\sqrt{2}) \right) \frac{d}{dx}.$$

Since $|m_\phi(0)| = \infty$ and $J_{s_\phi,m_\phi}(0) < \infty$, the end point 0 is $(s_\phi, m_\phi, 0)$ -exit. Since $|s_\phi(\infty)| = \infty$ and $J_{m_\phi,s_\phi}(\infty) = \infty$, the end point $\infty$ is $(s_\phi, m_\phi, 0)$ -natural.

**Example 3.7** Consider the generator

$$G = \frac{x^4}{2} \frac{d^2}{dx^2} + \frac{x^3}{2} \frac{d}{dx} - \frac{1}{8} x^2,$$

on $(0, \infty)$. The scale function, speed measure, and killing measure are given by

$$ds(x) = x^{-1}dx, \quad dm(x) = 2x^{-3}dx, \quad dk(x) = \frac{1}{4} x^{-1}dx.$$
The end points 0 and ∞ are \((s, m, k)\)-natural.

Set \(h(x) = x^{\frac{3}{2}}\). \(h(x)\) is a 0 harmonic function for \(\mathcal{G}\). We obtain the transformed scale function, the transformed speed measure, and the transformed generator as follows.

\[
\begin{align*}
    ds_h(x) &= x^{-2} \, dx, \\
    dm_h(x) &= 2x^{-2} \, dx, \\
    G_h &= \frac{x^4}{2} \frac{d^2}{dx^2} + x^3 \frac{d}{dx}.
\end{align*}
\]

Since \(|s_h(0)| = \infty\) and \(|m_h(0)| = \infty\), the end point 0 is \((s_h, m_h, 0)\)-natural. Since \(|s_h(\infty)| < \infty\) and \(|m_h(\infty)| < \infty\), the end point \(\infty\) is \((s_h, m_h, 0)\)-regular.

References


State of boundaries for harmonic transforms of one-dimensional generalized diffusion processes

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We consider a one-dimensional generalized diffusion operator $\mathcal{G}$ represented by triplet of Borel measures and a harmonic transform $\mathcal{G}_h$ of $\mathcal{G}$, where $h$ is a harmonic function for $\mathcal{G}$. Specially we treat an operator with killing measure which is not null measure. We consider the state of boundaries for the one-dimensional generalized diffusion process $\mathcal{D}_h$ with $\mathcal{G}_h$ as the generator. State of boundaries for $\mathcal{D}_h$ may be different from those for $\mathcal{D}$ which is a one-dimensional generalized process with $\mathcal{G}$ as the generator. We characterize the state of boundaries for $\mathcal{D}_h$ in terms of the Borel measures and a harmonic function for $\mathcal{G}$. After we prove our main theorem, we give some examples.