Linear estimates for the initial value problem of the fifth order equation

TOMOEDA Kyoko *

§1. Introduction

We consider the following initial value problem of the fifth order linear equation with an inhomogeneous term

$$
\begin{align*}
\partial_t u + \partial_x^5 u &= h(t, x) \quad t, x \in \mathbb{R}, \\
u(0, x) &= \phi(x) \quad x \in \mathbb{R}.
\end{align*}
$$

(1.1)

In this paper, we present some estimates for (1.1). These estimates play an essential role in the analysis on several nonlinear dissipative fifth order equations appearing in the study of long water wave, and in plasma physics: the fifth order KdV equation, the fifth order modified KdV equation and so on. In fact, we used in [6] these estimates to show the existence and uniqueness of time-local analytic solutions of the fifth order KdV type equation:

$$
\partial_t u + \partial_x^5 u = \partial_x (u^3) + \partial_x (\partial_x u)^2,
$$

which is obtained by removing $u\partial_x^3 u$ from the original fifth order KdV equation. S.Kwon also used in [5] some of these estimates without the proof to show the local well-posedness of the fifth order modified KdV equation (see Lemma 2.2 therein).

We first recall that, when $h \equiv 0$, the solution $u$ to (1.1) with the initial data $\phi$ in the Schwartz class is given by

$$
u(t, x) = e^{-it\partial_x^5} \phi(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{i\xi x} e^{i(x-y)\xi} \phi(y) dy d\xi.
$$

Due to Duhamel formula, the solution to the inhomogeneous linear equation (1.1) with $h$ in an appropriate class is then expressed as

$$
u(t, x) = e^{-it\partial_x^5} \phi(x) + \int_0^t e^{-i(t-t')\partial_x^5} h(t', x) dt'.
$$

(1.2)

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We next introduce a Bourgain space which accords with (1.1) (see [3]), and a smooth cut-off function: For $s, b \in \mathbb{R}$ define that

$$X_b^s = \{ f \in \mathcal{S}'(\mathbb{R}^2); \| f \|_{X_b^s} < \infty \},$$

where

$$\| f \|_{X_b^s}^s = \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |\tau + \xi|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\tau, \xi)|^2 d\tau d\xi,$$

and $\hat{f}$ is the Fourier transform of $f$ in both the $x$ and $t$ variables, that is,

$$\hat{f}(\tau, \xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} f(t, x) e^{-i\tau x - i\xi t} dt dx.$$

Let $\psi(t)$ be a cut-off function in $C_0^\infty(\mathbb{R})$ such that it takes the value 1 on the interval $[-1, 1]$ and the value 0 outside of $[-2, 2]$. We then put $\psi_T(t) = \psi(t/T)$.

Our main result now is stated as

**Theorem 1.1.** Let $s \in \mathbb{R}$, $b \in (1/2, 1)$, $a', a \in (0, 1/2)$ with $a' < a$, and let $0 < T < 1$. Then we have

$$\| \psi(t) e^{-t\partial_x^a} \phi(x) \|_{X_b^s} \leq C \| \phi \|_{H^s} \text{ for any } \phi(x) \in H^s(\mathbb{R}),$$

$$\left\| \psi(t) \int_0^t e^{-(t-t')\partial_x^a} h(t') dt' \right\|_{X_b^s} \leq C \| h \|_{X_{b-1}^s} \text{ for any } h(t, x) \in X_{b-1}^s(\mathbb{R}^2),$$

$$\| \psi_T h \|_{X_{s,a}^s} \leq CT^{(a-a')/4(1-a')} \| h \|_{X_{s,a}^s} \text{ for any } h(t, x) \in X_{s,a}^s(\mathbb{R}^2),$$

where $C > 0$ is a constant depending only on $s$, $b$, $a$ and $a'$.

We proceed the proof of Theorem 1.1 mainly as in the argument of Lemmas 3.1 and 3.3 in Kenig-Ponce-Vega [2] which studied the third order KdV equation. However, some modifications are needed to treat the fifth order operator $\partial_t + \partial_x^{5b}$ in the Bourgain space $X_b^s$ defined above.

Throughout we use the following notations: let $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ be the Fourier transform in the $x$ variable and the Fourier inverse transform in the $\xi$ variable; we define $\mathcal{F}_t$ and $\mathcal{F}_t^{-1}$ likewise; Riesz operators $D_x$ and $D_t$ are defined by $D_x = \mathcal{F}_x^{-1} |\xi| \mathcal{F}_x$ and $D_t = \mathcal{F}_t^{-1} |\tau| \mathcal{F}_t$; $L^p_t L^q_x$ denotes the space $L^p(\mathbb{R}_t; L^q(\mathbb{R}_x))$ for $1 \leq p, q \leq \infty$ with the norm defined by

$$\| f \|_{L^p_t L^q_x} = \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(t, x)|^p dt \right)^{q/p} dx \right)^{1/q}.$$

We also use the Sobolev space $H^s(\mathbb{R}) = \{ u \in \mathcal{S}'(\mathbb{R}); \langle D_x^s u \rangle \in L^2(\mathbb{R}) \}$ equipped with the norm $\| \cdot \|_{H^s(\mathbb{R})} = \| \langle D_x^s \cdot \rangle \|_{L^2(\mathbb{R})}$, where $\langle \cdot \rangle = (1 + |\cdot|)$. Any constant which may change from line to line is denoted by $C$. 

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§2. Proof of Theorem 1.1

Firstly we state a Leibniz rule for fractional derivatives (see Theorem 2.8 in Kenig-Ponce-Vega [2]).

Lemma 2.1. ([2]). Let \( 0 < \alpha < 1 \). Then we have
\[
\| D^\alpha (fg) - f(D^\alpha g) \|_{L^2} \leq c\| g \|_{L^\infty} \| D^\alpha f \|_{L^2},
\]
for any \( g \in L^\infty(\mathbb{R}) \) and \( f \in H^\alpha(\mathbb{R}) \).

Next we present an auxiliary estimate needed for the proof of Theorem 1.1.

Lemma 2.2. Let \( s \in \mathbb{R} \) and let \( 0 < b < 1 \). Then we have
\[
\| \psi(t)h \|_{X^s_b} \leq C \| h \|_{X^s_b} \quad \text{for any } h(t, x) \in X^s_b(\mathbb{R}^2),
\]
where \( C > 0 \) is a constant depending only on \( s \) and \( b \).

Proof. Fix \( x \in \mathbb{R} \). It is easy to see that we have only to show
\[
\int_\mathbb{R} (\tau + a)^{2b} |\mathcal{F}_t(\psi h)(\tau)|^2 d\tau \leq C \int_\mathbb{R} (\tau + a)^{2b} |\mathcal{F}_t h(\tau)|^2 d\tau \quad \text{for any } a \in \mathbb{R}.
\]

We first notice by the definition of \( (\tau + a)^{2b} \) that
\[
\int_\mathbb{R} (\tau + a)^{2b} |\mathcal{F}_t(\psi h)(\tau)|^2 d\tau \leq C \int_\mathbb{R} |\mathcal{F}_t(\psi h)(\tau)|^2 d\tau + C \int_\mathbb{R} (\tau + a)^{2b} |\mathcal{F}_t h(\tau)|^2 d\tau.
\]

Since \( \| \psi(t) \|_{L^\infty} \leq C \), we have by Plancherel's theorem
\[
\int_\mathbb{R} |\mathcal{F}_t(\psi h)(\tau)|^2 d\tau = C \int_\mathbb{R} |\psi(t)h(t)|^2 dt \leq C \| \psi(t) \|_{L^\infty}^2 \| h(t) \|_{L^2}^2 \leq C \| \mathcal{F}_t h(\tau) \|_{L^2}^2.
\]

Using Plancherel's theorem again, we also obtain
\[
\int_\mathbb{R} |\tau + a|^{2b} |\mathcal{F}_t(\psi h)(\tau)|^2 d\tau = C \| D^b_t(e^{iat} h(t)\psi(t)) \|_{L^2}^2 \\
\leq C \| D^b_t(e^{iat} h(t)\psi(t) - e^{iat} h(t)(D^b_t\psi(t))) \|_{L^2}^2 + C \| e^{iat} h(t)(D^b_t\psi(t)) \|_{L^2}^2 \\
\equiv I_1 + I_2.
\]

As for \( I_1 \), we apply Lemma 2.1 with \( \alpha = b \) to get
\[
I_1 \leq C \| \psi(t) \|_{L^\infty}^2 \| D^b_t(e^{iat} h(t)) \|_{L^2}^2 \leq C \| (\tau + a)^b \mathcal{F}_t h(\tau) \|_{L^2}^2.
\]

On the other hand, since \( \| D^b_t\psi(t) \|_{L^\infty} \leq C \), we obtain
\[
I_2 \leq C \| D^b_t\psi(t) \|_{L^\infty}^2 \| e^{iat} h(t) \|_{L^2}^2 \leq C \| e^{iat} h(t) \|_{L^2}^2 \| (\tau + a)^b \mathcal{F}_t h(\tau) \|_{L^2}^2.
\]

So combining (2.3)-(2.7), we reach the estimate (2.2). The proof of Lemma 2.2 is completed. \( \square \)
Now we are now in a position to give

Proof of Theorem 1.1. First we prove (1.3). Since

\[
(\psi(t)e^{-t\partial_x^\varepsilon}\phi(x)) = \mathcal{F}_t(\psi(t)e^{-t\xi^5}\mathcal{F}_x\phi(\xi)) = \mathcal{F}_t\psi(\tau + \xi^5)\mathcal{F}_x\phi(\xi),
\]

we calculate

\[
\|\psi(t)e^{-t\partial_x^\varepsilon}\phi(x)\|_{L^2_x}^2 = \|\langle \tau + \xi^5 \rangle^b \langle \xi \rangle^s \mathcal{F}_t\psi(\tau + \xi^5)\mathcal{F}_x\phi(\xi)\|_{L^2_t L^2_x}^2
\]

\[
= \int_\mathbb{R} \langle \xi \rangle^{2s} |\mathcal{F}_x\phi(\xi)|^2 \left( \int_\mathbb{R} \langle \tau + \xi^5 \rangle^{2b} |\mathcal{F}_t\psi(\tau + \xi^5)|^2 d\tau \right) d\xi. \tag{2.8}
\]

For the inner integral in the above, we can see

\[
\int_\mathbb{R} \langle \tau + \xi^5 \rangle^{2b} |\mathcal{F}_t\psi(\tau + \xi^5)|^2 d\tau \leq C \int_\mathbb{R} |\mathcal{F}_t\psi(\tau + \xi^5)|^2 d\tau + C \int_\mathbb{R} |\tau + \xi^5|^{2b} |\mathcal{F}_t\psi(\tau + \xi^5)|^2 d\tau
\]

\[
\leq C\|\psi(t)\|_{L^2_x}^2 + C\|\mathcal{D}_t^1 \psi(t)\|_{L^2_x}^2.
\tag{2.9}
\]

The desired estimate (1.3) readily follows from (2.8) and (2.9).

We now turn to the proof of (1.4). Noting that

\[
\mathcal{F}_x(e^{-(t-t')\partial_x^\varepsilon} h(t', x)) = e^{-i(t-t')\xi^5} \mathcal{F}_x h(t', \xi),
\]

and

\[
\int_0^t e^{i(t+\xi^5)\xi'} dt' = \left[ \frac{e^{i(t+\xi^5)\xi'}}{i(t + \xi^5)} \right]_0^t = \frac{e^{i(t+\xi^5)\xi'} - 1}{i(t + \xi^5)},
\]

and using Fubini’s theorem, we have

\[
\psi(t) \int_0^t e^{-(t-t')\partial_x^\varepsilon} h(t', x) dt' = \psi(t) \int_0^t \left\{ \int_{\mathbb{R}^2} e^{-i(t-t')\xi^5} \tilde{h}(\tau, \xi) e^{i\xi \varepsilon} e^{i\tau t} d\tau d\xi \right\} dt'
\]

\[
= \psi(t) \left\{ \int_{\mathbb{R}^2} \left( \int_0^t e^{i(t+\xi^5)\xi'} dt' \right) \tilde{h}(\tau, \xi) e^{i\xi \varepsilon} e^{-i\xi^5 \tau} d\tau d\xi \right\}
\]

\[
= \psi(t) \left\{ \int_{\mathbb{R}^2} \frac{e^{i(t+\xi^5)\xi'} - 1}{i(t + \xi^5)} \tilde{h}(\tau, \xi) e^{i\xi \varepsilon} e^{-i\xi^5 \tau} d\tau d\xi \right\}
\]

\[
= \psi(t) \int_{\mathbb{R}} e^{i\xi \varepsilon} \tilde{h}(\tau, \xi) \left\{ \int_{\mathbb{R}} \frac{e^{i(t+\xi^5)\xi'} - 1}{i(t + \xi^5)} \tilde{h}(\tau, \xi) \psi(\tau + \xi^5) d\tau \right\} d\xi
\]

\[
+ \psi(t) \int_{\mathbb{R}} e^{i\xi \varepsilon} \tilde{h}(\tau, \xi) [1 - \psi(\tau + \xi^5)] \frac{e^{i\tau t} - e^{-i\xi^5}}{i(t + \xi^5)} d\tau d\xi
\]

\[
\equiv I_1 + I_2. \tag{2.10}
\]
Regarding $I_1$, we first use the Taylor expansion to obtain

$$I_1 = \sum_{k=1}^{\infty} \frac{t^k}{k!} \psi(t) \left\{ \int_R e^{i(x \xi - t \xi^3)} \left( \int_R \hat{h}(\tau, \xi) \psi(\tau + \xi^5)(\tau + \xi^5)^{k-1} d\tau \right) d\xi \right\}.$$

(2.11)

On the other hand, setting

$$t^k \psi(t) = \psi_k(t), \quad k \geq 1,$$

and using Plancherel’s theorem and the assumptions $b < 1$, we find

$$\int_R |\mathcal{F}_t \psi_k(\tau)|^2 (1 + |\tau|)^{2b} d\tau \leq C \int_R |\mathcal{F}_t \psi_k(\tau)|^2 d\tau + C \int_R |\tau|^2 |\mathcal{F}_t \psi_k(\tau)|^2 d\tau$$

$$= C \int_R |\psi_k(t)|^2 d\tau + C \int_R |D_t \psi(t)|^2 d\tau$$

$$\leq C \int_{-2}^{2} |t|^2 dt + C \int_{-2}^{2} |kt^{k-1} + t^k|^2 dt$$

$$\leq 2^{2k} C(k+1)^2.$$  (2.12)

Therefore, a similar argument to the proof of (1.3) together with (2.12) yields

$$\|I_1\|_{X_b}^2 \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left\| \psi_k(t) \left\{ \int_R e^{i(x \xi - t \xi^3)} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)(\tau + \xi^5)^{k-1} d\tau \right) d\xi \right\} \right\|_{X_b}^2$$

$$= C \sum_{k=1}^{\infty} \frac{1}{k!} \left\| \psi_k(t) e^{-i\theta_2} \mathcal{F}^{-1}_{\xi} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1}(\tau + \xi^5) d\tau \right) \right\|_{X_b}^2$$

$$\leq \sum_{k=1}^{\infty} \frac{2^{2k} (k+1)^2}{k!} \left\| \mathcal{F}^{-1}_{\xi} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1}(\tau + \xi^5) d\tau \right) \right\|_{H^s}^2.  \quad (2.13)$$

Here we used the fact that

$$\int_R e^{i(x \xi - t \xi^3)} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1}(\tau + \xi^5) d\tau \right) d\xi$$

$$= \mathcal{F}^{-1}_{\xi} \left( e^{-i\theta_5} \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1}(\tau + \xi^5) d\tau \right)$$

$$= e^{-i\theta_5} \mathcal{F}^{-1}_{\xi} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1}(\tau + \xi^5) d\tau \right).$$
Then, applying Schwarz's inequality with respect to $\tau$, and noting $b > \frac{1}{2}$, we have

$$
\sum_{k=1}^{\infty} \frac{2^k (k+1)^2}{k!} \left\| \mathcal{F}_\xi^{-1} \left( \int_R \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1} \psi(\tau + \xi^5) d\tau \right) \right\|_{H^s}^2
\leq C \sum_{k=1}^{\infty} \frac{2^k (k+1)^2}{k!} \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_{|\tau+\xi^5|<2} \hat{h}(\tau, \xi)(\tau + \xi^5)^{k-1} d\tau \right)^2 d\xi \right\}
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_{|\tau+\xi^5|<2} |\hat{h}(\tau, \xi)| d\tau \right)^2 d\xi \right\}
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_R \frac{|\hat{h}(\tau, \xi)|^2}{(1 + |\tau + \xi^5|)^{1-b}} \cdot \frac{1}{(1 + |\tau + \xi^5|)^b} d\tau \right)^2 d\xi \right\}
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_R \frac{|\hat{h}(\tau, \xi)|^2}{(1 + |\tau + \xi^5|)^{2s(1-b)}} \cdot \frac{1}{(1 + |\tau + \xi^5|)^{2b}} d\tau \right) d\xi \right\}
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_R |\hat{h}(\tau, \xi)|^2 d\tau \right) d\xi \right\} = C \|h\|_{X^{s-1}}^2
$$

and so

$$
\|I_1\|_{X^s} \leq C \|h\|_{X^{s-1}}.
$$

Finally we give the estimate of the term $I_2$ in (2.10). First we divide $I_2$ into two parts:

$$
I_2 = I_{2.1} + I_{2.2},
$$

where

$$
I_{2.1} = -\psi(t) \int_R e^{i(x\xi - t\xi^5)} \left( \int_R \frac{1 - \psi(\tau + \xi^5)}{i(\tau + \xi^5)} \hat{h}(\tau, \xi) d\tau \right) d\xi,
I_{2.2} = \psi(t) \int_{R^2} e^{i(x\xi + t\tau)} \frac{1 - \psi(\tau + \xi^5)}{i(\tau + \xi^5)} \hat{h}(\tau, \xi) d\tau d\xi.
$$

Regarding $I_{2.1}$, a similar argument to (2.14) leads to

$$
\|I_{2.1}\|_{X^s}^2 \leq C \left\| \mathcal{F}_\xi^{-1} \left( \int_R \frac{1 - \psi(\tau + \xi^5)}{i(\tau + \xi^5)} \hat{h}(\tau, \xi) d\tau \right) \right\|_{H^s}^2
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_{|\tau+\xi^5|<2} |\hat{h}(\tau, \xi)| d\tau \right)^2 d\xi \right\}
\leq C \left\{ \int_R (1 + |\xi|)^{2s} \left( \int_R \frac{|\hat{h}(\tau, \xi)|^2}{(1 + |\tau + \xi^5|)^{1-b}} \cdot \frac{1}{(1 + |\tau + \xi^5|)^b} d\tau \right)^2 d\xi \right\}
\leq C \|h\|_{X^{s-1}}^2.
$$

(2.15)
As for $I_{2,2}$, according to Lemma 2.2, we have

$$
\|I_{2,2}\|_{X^s}^2 \leq C \left\| \int_{\mathbb{R}^2} e^{ix(t+\tau+\xi)} \frac{1 - \psi(t+\xi^5)}{(t+\xi^5)^{2b}} \hat{h}(\tau,\xi) d\tau d\xi \right\|_{X^s}^2
$$

$$
\leq C \left\{ \int_{\mathbb{R}^2} (1 + |\tau + \xi^5|)^{2b} \left( 1 + |\xi| \right)^2 \left| \hat{h}(\tau,\xi) \right|^2 \frac{1 - \psi(t+\xi^5)}{\tau + \xi^5} d\tau d\xi \right\}
$$

$$
\leq C \left\{ \int_{\mathbb{R}} (1 + |\xi|)^{2s} \left( \int_{1<|\tau+\xi^5|} (1 + |\tau + \xi^5|)^{2(b-1)} \frac{\left| \hat{h}(\tau,\xi) \right|^2}{(1 + |\tau + \xi^5|)^2} d\tau \right) d\xi \right\}
$$

$$
\leq C \left\{ \int_{\mathbb{R}^2} (1 + |\tau + \xi^5|)^{2(b-1)} (1 + |\xi|)^{2s} \left| \hat{h}(\tau,\xi) \right|^2 d\tau d\xi \right\}
$$

$$
\leq C \|h\|_{X^s_{-1}}^2. \quad (2.16)
$$

Now it follows from (2.15) and (2.16) that

$$
\|I_2\|_{X^s} \leq C \|h\|_{X^s_{-1}}.
$$

Consequently, we have established the estimate (1.4). As for the proof of (1.5), see Lemma 2.5 in Ginibre-Tsutsumi-Velo [4]. The proof of the Theorem 1.1 is completed.

References


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\end{cases}
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\[
\partial_t u + \partial_x^5 u = \partial_x(u^3) + \partial_x(\partial_x u)^2,
\]

which is obtained by removing \( u\partial_x^3 u \) from the nonlinear part of the original fifth order KdV equation.

We proceed the proof of Theorem 1.1 mainly as in the argument of Lemmas 3.1 and 3.3 in Kenig-Ponce-Vega [2] which studied the third order KdV equation. However, some modifications are needed to treat the fifth order operator \( \partial_t + \partial_x^5 \) in a certain Bourgain space \( X_6^s \).