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Numerical approximation of weak solutions for boundary value problems
by
Reiko Sakamoto

Introduction. There are a lot of existence theorems for boundary value problems relating to linear partial differential equations. Formulations of boundary value problems are different according to types of differential operators, i.e. elliptic, parabolic, hyperbolic and etc.. But their methods of proof seem very similar. The most universal method owes to Riesz' Theorem in some Hilbert space, based on some energy estimates. Contrary to general methods of existence theorems, methods of numerical approximation of solutions seem strictly combined with some positive forms(e.g.[1]). Since Riesz' Theorem itself proves existence of solutions in abstract sense, it seems that there is a very long distance between existence and approximation of solutions. Our aim in this paper is to see that the method of approximation is just behind Riesz' Theorem.

In this paper, weak solutions are defined by using supplementary functions, which play essential role for approximation of solutions. Existence of weak solutions is proved by using Riesz' Theorem, based on a weak energy estimate on the adjoint problem. The essence of this idea is found in [2],[3],etc. Concerning to approximation, the trigonometrical functions are used as basis functions. It is the most remarkable point that basis functions can be chosen without any consideration of domains or boundary conditions.

Our problem is as follows. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Let \( \Gamma \) be the boundary of \( \Omega \), which is a finite sum of smooth surfaces, i.e. \( \Gamma = \bigcup_{i\in I} \Gamma_i \), where \( \Gamma_i \) is an open manifold of dimension \( n-1 \). Let us consider a boundary value problem:

\[
\begin{align*}
& \text{Au=f in } \Omega, \\
& B_{i}^{(1)} u = 0 \text{ on } \Gamma_i \ (j=1,\ldots,b, i=1,\ldots,h),
\end{align*}
\]

where \( \{A;B_{i}^{(1)}\} \) are linear partial differential operators, \( f \) is a given function of \( L^2(\Omega) \), and \( u \) is an unknown function of \( L^2(\Omega) \).

The following Assumption(A) is assumed throughout this paper.

Assumption(A).
1) Let \( x \in \overline{\bigcap \cap \cdots \cap \Gamma_1 \Gamma_2 \ldots \Gamma_h} \) and \( x \in \overline{\bigcup \cup \cdots \cup \Gamma_1 \Gamma_2 \ldots \Gamma_h} \), then there exists \( U(x) \) (a neighbourhood of \( x \)) such that

\[
\begin{align*}
\Phi(U(x)) &= V \\
\Phi(\Omega \cap U(x)) &= \{ y \in V \mid y_1 > 0, y_2 > 0, \ldots, y_r > 0 \},
\end{align*}
\]
\[ \Phi(\Gamma_{\gamma} \cap U(x)) = \{ y \in V \mid y_1 = 0, y_2 > 0, \ldots, y_{\gamma} > 0 \}, \]
\[ \Phi(\Gamma_{\gamma} \cap U(x)) = \{ y \in V \mid y_1 > 0, y_2 = 0, y_3 > 0, \ldots, y_{\gamma} > 0 \}, \]
\[ \ldots \]
\[ \Phi(\Gamma_{\gamma} \cap U(x)) = \{ y \in V \mid y_{\gamma} > 0, \ldots, y_{\gamma} = 0 \}, \]
where \( V = \{ y \in \mathbb{R}^n \mid |y| < 1 \} \). We say that \( x \in \Gamma \) is a \( \gamma \)-ple point of \( \Gamma \), if \( x \in \Gamma_{\gamma} \cap \ldots \cap \Gamma_{\gamma} \) and \( x \in \Gamma_{\gamma} \cap \Gamma_{\gamma} \cup \ldots \cup \Gamma_{\gamma} \), where \( 1 \leq \gamma \leq n \).

2) Let \( A = A(x, D_x) \) be a linear partial differential operator of order \( m \) with smooth coefficients. Let \( A(x, D_x) \) be the principal part of \( A \). We assume that \( A \) is normal on \( \Gamma_1 \), that is, \( |A(x, n^{(1)}(x))| > c > 0 \) \( (x \in \Gamma_1) \), where \( n^{(1)}(x) \) is the unit exterior normal vector at \( x \in \Gamma_1 \).

3) Let \( B_j^{(1)} = B_j^{(1)}(x, D_x) \) be a linear partial differential operator of order \( m_j^{(1)} \) \( (0 \leq m_j^{(1)} \leq m-1) \). Let \( B_j^{(1)}(x, D_x) \) be the principal part of \( B_j^{(1)} \). We assume that \( \{ B_j^{(1)}(j=1, \ldots, b^{(1)}) \} \) is a normal set on \( \Gamma_1 \), that is, \( |B_j^{(1)}(x, n^{(1)}(x))| > c > 0 \) \( (x \in \Gamma_1) \) and \( m_j^{(1)} \neq m_k^{(1)} \) \( (j \neq k) \).

We use notations:

\[ \| \cdot \| = \| \cdot \|_{L^2(\Omega)}, \quad (\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)} \]
\[ \| \cdot \|_\sigma = \| \cdot \|_{H^\sigma(\Omega)}, \quad (\cdot, \cdot)_\sigma = (\cdot, \cdot)_{H^\sigma(\Omega)} \]
\[ \langle \cdot, \cdot \rangle_1 = \| \cdot \|_{L^2(\Gamma_1)}, \quad \langle \cdot, \cdot \rangle_1 = (\cdot, \cdot)_{L^2(\Gamma_1)} \]
\[ \langle \cdot, \cdot \rangle_1, \sigma = \| \cdot \|_{H^\sigma(\Gamma_1)}, \quad \langle \cdot, \cdot \rangle_1, \sigma = (\cdot, \cdot)_{H^\sigma(\Gamma_1)} \]
where \( \sigma \) is a non-negative integer.

Remark. \( \langle \cdot, \cdot \rangle_1, \sigma \) and \( \langle \cdot, \cdot \rangle_1, \sigma \) may be used for fractional \( \sigma \), in case when \( \Gamma_1 \) is closed.

\section{Existence of weak solutions.}

Let
\[ A(x, D_x) = \sum a_\nu(x) D_x^{\nu} \quad (D_x^{\nu} = i^{\nu} \partial x^{\nu}), \]
\[ |\nu| \leq m \]
and
\[ A^*(x, D_x) = \sum D_x^{\nu} a_\nu(x) \quad (|\nu| \leq m) \]
Let \( \{ m_j^{(1)} (j=b^{(1)}+1, \ldots, m) \} \) be defined such as
\{m_j^{(1)} (j=1,2,...,b^{(1)})\} \cup \{m_j^{(1)} (j=b^{(1)}+1,...,m)\} = \{0,1,...,m-1\},

and define
\[
m_j^{(1)} (x,D_x) = (d/dn)^{(1)} (j=b^{(1)}+1,...,m),
\]

Then \{B_j^{(1)} (x,D_x) (j=1,...,m)\} becomes also a normal set on \(\Gamma_i\) of orders \(\{0,1,...,m-1\}\). Corresponding to \{B_j^{(1)} (x,D_x) (j=1,...,m)\}, as is well known, there exists a normal set of partial differential operators \{B_j^{(1)} (x,D_x) (j=1,...,m)\} near \(\Gamma_i\), whose differential orders are \(\{m_j^{(1)} = m-1-m_j^{(1)}\}\). Then it is well known.

**Lemma 1.1.** Suppose that \(u,f \in L^2(\Omega)\) satisfy \(Au = f\) in \(\Omega\) (dis). Then \(u\) has traces \(B_j^{(1)} u \in \mathcal{D}'(\Gamma_i)\) and it holds
\[
(G) \quad (f,v) - (u,A^*v) = \sum_{1 \leq j \leq m} \langle B_j^{(1)} u, B_j^{(1)} v \rangle^{(1)}
\]

for any \(v \in C^\infty(\overline{\Omega})\), satisfying \(\text{supp}[v] \cap \Gamma = \text{supp}[v] \cap \Gamma_i\), where \(\langle , \rangle^{(1)}\) is interpreted as the duality of \(\mathcal{D}'(\Gamma_i)\) and \(\mathcal{D}(\Gamma_i)\).

Now we define the adjoint problem \((P')\) corresponding to \((P)\) by
\[
(P') \begin{cases}
A^*v = f' & \text{in } \Omega, \\
B_j^{(1)} v = g_j^{(1)} & \text{on } \Gamma_i (j=b^{(1)}+1,...,m, i=1,...,h),
\end{cases}
\]

where we assume
\[
(E') \quad \|v\| = C\{ \|A^*v\|^2 + \sum_{i=1}^{h} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} v, B_j^{(1)} v \rangle^{(1)} \}
\]

for any \(v \in H^m(\Omega)\),

where \(M = \max\{m, m_i^{(1)} + \sigma_{ij}^{(1)} + 1\}\).

Let \(H\) be a Hilbert space defined by the completion of \(H^m(\Omega)\) with respect to the norm
\[
\|v\| = \|A^*v\|^2 + \sum_{i=1}^{h} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} v, B_j^{(1)} v \rangle^{(1)}.
\]

Inner product of \(H\) is defined by

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\[ [w,v] = (A^*w, A^*v) + \sum_{i=1}^{h} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} w, B_j^{(1)} v \rangle \cdot \sigma_{1i} \]

Remark. Weak energy estimate \((E')\) means that
\[ \|v\| \leq C \|v\| \quad (\forall v \in H). \]

Our reasoning depends on the following well known theorem.

**Riesz' Theorem**: Let \( \hat{\lambda}[v] \) be a continuous anti-linear functional in \( H \).
Then there exists \( w \in H \) such that
\[ \hat{\lambda}[v] = [w,v] \]
and
\[ \|w\| = \sup_{v \in H} \frac{\|\hat{\lambda}[v]\|}{\|v\|}. \]
We say that \( w \) is a **Riesz' function** of \( \hat{\lambda}[v] \).

Let \( f \in L^2(\Omega) \), then we have
\[ |(f,v)| \leq \|f\| \|v\| \leq C \|f\| \|v\| \quad (\forall v \in H), \]
therefore \( \hat{\lambda}[v]=(f,v) \) defines a continuous anti-linear functional in \( H \).
Owing to Riesz' Theorem, there exists \( w \in H \) such that
\[ (f,v) = [w,v] \quad (\forall v \in H), \quad \|w\| \leq C \|f\|. \]
It means that
\[ (f,v) = (A^*w, A^*v) + \sum_{i=1}^{h} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} w, B_j^{(1)} v \rangle \cdot \sigma_{1i}, \quad (\forall v \in H). \]

Set \( u = A^*w \), then
\[ (f,v) - (u, A^*v) = \sum_{i=1}^{h} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} w, B_j^{(1)} v \rangle \cdot \sigma_{1i}, \quad (\forall v \in H). \]

Hence we have
1) \( (f, \phi) - (u, A^*\phi) = 0 \quad (\forall \phi \in \mathcal{D}(\Omega)), \)
that is, \( Au=f \) in \( \Omega \) (dis.).
2) \( \langle B_j^{(1)} u, \phi \rangle_{(1)} = 0 \quad (\forall \phi \in \mathcal{D}(\Gamma_i)), \)
that is, \( B_j^{(1)} u = 0 \) on \( \Gamma_i \) (dis.) \((j=1, \ldots, b^{(1)})\).

**Proof of 2**. Let \( v \in \mathcal{C}^\infty(\Omega) \) such that \( \text{supp}[v] \cap \Gamma = \text{supp}[v] \cap \Gamma_i \)
then
holds. On the other hand, we have
\[(f,v)-(u,A^*v) = \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} w, B_j^{(1)} v \rangle_{(1)} . \sqrt{\gamma} \]
\[1 \leq j \leq m \]
from (G). Hence we have
\[\sum_{j=1}^{m} \langle B_j^{(1)} u, B_j^{(1)} v \rangle_{(1)} = \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} w, B_j^{(1)} v \rangle_{(1)} . \sqrt{\gamma} \]
\[(\forall v \in C^m(\overline{\Omega})) , \]
which means \(B_j^{(1)} u=0\) on \(\Gamma_1\) (dis.) \((j=1,..,b^{(1)})\). □

We say that \(u \in L^2(\Omega)\) is a weak solution of (P), if
\[Au=f \quad \text{in } \Omega \text{ (dis.),} \]

\[(P)_{\alpha} \quad \left\{ \begin{array}{l} B_j^{(1)} u=0 \quad \text{on } \Gamma_1 \text{ (dis.)} \quad (j=1,..,b^{(1)}, i=1,..,h) \end{array} \right. \]
is satisfied. We say that \(u \in L^2(\Omega)\) is a \(H\)-weak solution of (P), if \(u=A^*w\), where \(w \in H\) satisfies
\[\langle w,v \rangle = (f,v) \quad \forall v \in H. \]
We call \(w\) a supplementary function of \(H\)-weak solution of (P). A supplementary function of \(H\)-weak solution of (P) is a Riesz'function of \(Q[\cdot,\cdot]=(f,\cdot)\). As is shown above, \(H\)-weak solution of (P) is a weak solution of (P). Here we have

**Theorem I.** Assume \((E')\). Then for any \(f \in L^2(\Omega)\), there exists a unique \(H\)-weak solution \(u \in L^2(\Omega)\) of (P) and it holds
\[\|u\| \leq C \|f\|, \]
where \(C\) is independent of \(f\).

Let us consider a generalization of Theorem I. Namely, let us consider a solution \(u \in L^2(\Omega)\) for \(f \in (H^\ast(\Omega))^\prime\), where \((H^\ast(\Omega))^\prime\) is the dual space of \(H^\ast(\Omega)\) with dual norm
\[\|f\|_{\ast} = \sup_{v \in H^\ast(\Omega), \|v\|_{\ast}} \|(f,v)\|, \]
where \((\cdot,\cdot)\) is interpreted as the duality of \((H^\ast(\Omega))^\prime\) and \(H^\ast(\Omega)\).

Let us assume
\[(E'_{\ast}) \quad \|v\|_{\ast} \leq C\{\|A^*v\|^2 + \sum_{i=1}^{m} \sum_{j=b^{(1)}+1}^{m} \langle B_j^{(1)} v, B_j^{(1)} v \rangle_{(1)} . \sqrt{\gamma} \}
\[\forall v \in H^\ast(\Omega) , \]

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instead of \((E')\), then it holds
\[
\|v\| \leq C \|v\| \quad (\forall v \in H^m(\Omega)),
\]
where \(0 \leq \mu \leq m\).

Let \(f \in (H^m(\Omega))'\), then we have
\[
|(f,v)| \leq \|f\|_{-\mu} \|v\| \leq C \|f\|_{-\mu} \|v\| \quad (\forall v \in H).
\]
Hence, as in the above, there exists \(w \in H\) such that
\[
(f,v) = [w,v] \quad (\forall v \in H), \quad \|w\| \leq C \|f\|_{-\mu},
\]
owing to Riesz' Theorem. Here we have

**Theorem II.** Assume \((E')_\mu\). Then for any \(f \in (H^m(\Omega))'\), there exists a unique \(H\)-weak solution \(u \in L^2(\Omega)\) of \((P)\) and it holds
\[
\|u\| \leq C \|f\|_{-\mu}.
\]

**Example 1.** Let \(\Omega\) be a domain in \(\mathbb{R}^2\), bounded by two closed curves \(\Gamma_1\) and \(\Gamma_2\) without any intersection. Let us consider
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
\Gamma_1 : u &= \phi \quad \text{on } \Gamma_1, \\
\Gamma_2 : (d/dn)u &= 0 \quad \text{on } \Gamma_2,
\end{align*}
\]
where \(\Delta = \partial_x^2 + \partial_y^2\). Choose \(\Phi\) such that \(\Phi = \phi\) on \(\Gamma_1\), \(\Phi = 0\) near \(\Gamma_2\),

and set
\[
u = u - \Phi, \quad f = -\Delta \Phi,
\]
then \((P)\) is reduced to
\[
\begin{align*}
\Delta u &= f \quad \text{in } \Omega, \\
\Gamma_1 : u &= 0 \quad \text{on } \Gamma_1, \\
\Gamma_2 : (d/dn)u &= 0 \quad \text{on } \Gamma_2.
\end{align*}
\]
The adjoint problem of \((P)\) is
\[
\begin{align*}
\Delta v &= f' \quad \text{in } \Omega, \\
\Gamma_1 : v &= g \quad \text{on } \Gamma_1, \\
\Gamma_2 : (d/dn)v &= h \quad \text{on } \Gamma_2.
\end{align*}
\]
Since we know an energy inequality for \((P')\):
\[
\|v\|^2 \leq C \{ \|\Delta v\|^2 + \langle v, v \rangle_{(1), 1/2} + (d/dn)v, v \rangle_{(2), 1/2} \} \quad (\forall v \in H^2(\Omega)),
\]
we may define \(H\) as the completion of \(H^2(\Omega)\) by the norm:
\[
\|v\|^2 = \|\Delta v\|^2 + \langle v, v \rangle_{(1), 1/2} + (d/dn)v, v \rangle_{(2), 1/2}.
\]

**Example 2.** Let \(\omega\) be a domain in \(\mathbb{R}^2\) set
\[
\Omega = (0,T) \times \omega, \quad \Gamma_0 = (0,T) \times \partial \omega, \quad \Gamma_1 = \{t=0\} \times \omega, \quad \Gamma_2 = \{t=T\} \times \omega,
\]
and consider
\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t^2 - \Delta \right) U = 0 \quad \text{in } \Omega, \\
U = 0 \quad \text{on } \Gamma_0, \\
U = \phi \quad \text{and} \quad \partial_t U = \phi_1 \quad \text{on } \Gamma_1,
\end{array} \right.
\]
where \(\Delta = \partial_x^2 + \partial_y^2\), and \(\phi_0 = \phi_1 = 0\) on \(\partial \Gamma_1\). Set
\[
\Phi = \phi_0 + t \phi_1, \quad u = U - \Phi, \quad f = \Delta \Phi,
\]
then \((P)\) is reduced to
\[
\begin{align*}
\{ (\partial_t^2 - \Delta)u &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \Gamma^0, \\
u &= \partial_n u = 0 & \text{on } \Gamma_1.
\end{align*}
\]

The adjoint problem of (P) is
\[
\begin{align*}
(\partial_t^2 - \Delta)v &= f' & \text{in } \Omega, \\
v &= g & \text{on } \Gamma^0, \\
v &= h_0, \quad \partial_t v = h_1 & \text{on } \Gamma_1.
\end{align*}
\]

Since we know an energy inequality for (P'):
\[
\|v\| \leq C\{ \| (\partial_t^2 - \Delta)v \| + \langle v \rangle \|_{(1)} + \langle v \rangle \|_{(2)} + \langle \partial_t v \rangle \|_{(2)} \} \qquad (\forall v \in H^2(\Omega)),
\]
we may define \(H\) as the completion of \(H^2(\Omega)\) by the norm:
\[
\|v\|^2 = \| (\partial_t^2 - \Delta)v \|^2 + \langle v \rangle \|_{(1)}^2 + \langle v \rangle \|_{(2)}^2 + \langle \partial_t v \rangle \|_{(2)}^2.
\]

\section{2. Approximation of weak solutions.}

**Lemma 2.1.** Suppose that \(\text{diam}(\Omega) < a\pi\). Set
\[
\Omega' = \bigcap_{j=1}^n (x_j^0 - a\pi, x_j^0 + a\pi)
\]
for fixed \(x^0 \in \Omega\). Then \(\overline{\Omega} \subset \Omega'\) and there exists a continuous linear map \(L_k\) from \(H^k(\Omega)\) to \(H^k(\Omega')\) such that
\[
(L_k w)(x) = w(x) \text{ in } \Omega, \quad \text{supp} [L_k w] \subset \Omega'.
\]

**Proof.** Since \(\overline{\Omega} \subset \Omega'\), we can choose \(\delta > 0\) such that
\[
\Omega \subset \Omega' \subset \Omega',
\]
where \(\Omega'\) is the \(\delta\)-neighbourhood of \(\Omega\). From Assumption(A-1), there exist \(x^{(1)}, \ldots, x^{(J)} \in \Gamma\) and their neighbourhoods \(U_1, \ldots, U_J\) such that
\[
\Gamma \subset \bigcup_{j=1}^J U_j, \quad \text{diam}(U_j) < \delta,
\]

\[
\Phi_j(U_j) = V, \quad \Phi_j(U_j \cap \Omega) = V \cap \Sigma_j,
\]

where \(\Sigma_j = \{ y \in \mathbb{R}^n \mid y > 0, \ldots, y_J > 0 \}\), where \(x^{(j)}\) is a \(\gamma\)-ple point of \(\Gamma\). Let \(\beta_j(x)\) be smooth functions such that
\[
\supp [\beta_j(x)] \subset U_j, \quad \sum_{j=1}^J \beta_j(x)^2 = 1 \text{ near } \Gamma.
\]

Set
\[
\beta_0(x) = \left\{ \begin{array}{ll}
1 - \sum_{j=1}^J \beta_j(x)^2 & (x \in \Omega), \\
0 & (x \in \Omega^c),
\end{array} \right.
\]
then
\[
W(x) = \sum_{j=1}^{J} \beta_j(x)^2 w(x) + \beta_0(x) w(x).
\]
Set
\[
W_j(y) = \begin{cases}
(\beta_j(x)^2 w(x)(\Phi^{-1}(y_j))) & (y \in \mathcal{V} \cap \Sigma_j), \\
0 & (y \in \mathcal{V}^* \cap \Sigma_j),
\end{cases}
\]
then
\[
\|W_j\|_{H^k(\Sigma_j)} \leq C_k \|w\|_{H^k(\mathcal{V})}.
\]
Let \(\{c_1, \ldots, c_k\}\) be defined by
\[
k \sum_{s=1}^{k} c_s (-s)^r = 1 \quad (r=0,1, \ldots, k-1).
\]
Set
\[
W_j^{(1)}(y) = \begin{cases}
W_j(y) & (y_1 > 0), \\
\sum_{s=1}^{k} c_s W_j(-s y_1, y_2, \ldots, y_n) & (y_1 < 0),
\end{cases}
\]
\[
W_j^{(2)}(y) = \begin{cases}
W_j^{(1)}(y) & (y_1 > 0), \\
\sum_{s=1}^{k} c_s W_j^{(1)}(y_1, -s y_1, y_2, \ldots, y_n) & (y_1 < 0),
\end{cases}
\]
and
\[
W_j^{(y)}(y) = W_j^{(1)}(y).
\]
Then we have
\[
W_j^{(y)}(y) = W_j(y) \quad (y \in \Sigma_j), \quad \text{supp}[W_j^{(y)}(y)] \subset \mathcal{V},
\]
and
\[
\|W_j^{(y)}\|_{H^k(R^n)} \leq C_k \|W_j\|_{H^k(\Sigma_j)}.
\]
Finally, set
\[
W^*(x) = \sum_{j=1}^{J} \beta_j(x) W_j^{(y)}(\Phi_j(x)) + \beta_0(x) w(x),
\]
then we have
\[
\text{supp}[W^*] \subset \Omega \epsilon, \quad W^*(x) = w(x) \quad (x \in \Omega),
\]
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and
\[ \| w \|_{H^k(\Omega')} \leq C_k \| w \|_{H^k(\Omega)}. \]

Hence we have a continuous linear map: \( w \to \tilde{w} \) from \( H^k(\Omega) \) to \( H^k(\Omega') \). \( \square \)

The following Lemma is well known in the theory of Fourier series.

**Lemma 2.2.** Let \( w \in H^k(\Omega') \), and set
\[ C_* = (2a\pi)^{-n} \langle w, \exp(i\alpha^{-1}\alpha \cdot x) \rangle \quad (\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n), \]
then
\[ \| w \|_{L^2(\Omega')} = \sum_{|\nu| \leq k} \| D^\nu w \|_{L^2(\Omega')} \]
\[ = (2a\pi)^n \sum_{\alpha \in \mathbb{Z}^n} \left\{ \sum_{|\nu| \leq k} (a^{-1}\alpha)^{\nu} \right\} |C_*|^{\nu} \]
and
\[ w(x) = \sum_{\alpha \in \mathbb{Z}^n} C_* \exp(i\alpha^{-1}\alpha \cdot x) \quad \text{in } H^k(\Omega'), \]
that is,
\[ \sum_{\alpha \in \mathbb{Z}^n} C_* (a^{-1}\alpha)^{\nu} \exp(i\alpha^{-1}\alpha \cdot x) \to D^\nu w \quad \text{in } L^2(\Omega') \quad (|\nu| \leq k) \]
\[ |\alpha| \leq N \]
as \( N \to \infty \).

From Lemma 2.1 and Lemma 2.2, we have

**Lemma 2.3.** Suppose that \( \text{diam}(\Omega) < a\pi \) and \( w \in H^m(\Omega) \). Then there exists
\[ \{ C_* \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \} \]
such that
\[ c_1 \| w \|_{H^m(\Omega)} \leq \sum_{|\alpha| \leq N} (1 + |\alpha|)^m |C_*| \leq c_2 \| w \|_{H^m(\Omega)} \]
and
\[ w(x) = \sum_{\alpha \in \mathbb{Z}^n} C_* \exp(i\alpha^{-1}\alpha \cdot x) \quad \text{in } H^m(\Omega). \]

Therefore, set
\[ w(x) = \sum_{|\alpha| \leq N} C_* \exp(i\alpha^{-1}\alpha \cdot x) \quad (x \in \Omega), \]
then it holds
\[ w_N \to w \quad \text{in } H \text{ as } N \to \infty. \]
Proof. Let \( w \in H^m(\Omega) \), and set \( w \sim = L_{\omega} w \in H^m(\Omega') \), where \( L_{\omega} \) is in Lemma 2.1. Let us apply Lemma 2.2 to \( w \sim \), then there exist \( \{C_\alpha\} \) such that

\[
\sum (1 + |\alpha|^2) |C_\alpha|^2 \leq C \|w\| \quad \text{and} \quad \|w\| \leq C' \|w\|
\]

\( \alpha \in \mathbb{Z}^n \) \( H^m(\Omega') \) \( H^m(\Omega) \)

and

\[
w\sim(x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha \exp(i\alpha \cdot x) \quad \text{in} \quad H^m(\Omega').
\]

Set

\[
w_N \sim(x) = \sum_{|\alpha| \leq N} C_\alpha \exp(i\alpha \cdot x), \quad w_N = w_N \sim |_\mathcal{Q},
\]

then

\[
\|w_N - w\| \Rightarrow \|w_N \sim - w\| \quad \Rightarrow 0 \quad \text{as} \quad N \to \infty,
\]

\( H^m(\Omega) \) \( H^m(\Omega') \)

therefore

\[
\|w_N - w\| \Rightarrow 0 \quad \text{as} \quad N \to \infty.
\]

Now we say that \( V = \{v_i \in H \mid (i=1,2,\ldots)\} \) is a set of basis functions of \( H \), if \( \text{sp} \ V \), the linear space spanned by \( V \), is dense in \( H \). Since \( H^m(\Omega) \) is dense in \( H \), we have from Lemma 2.3.

Proposition 2.1. Let \( \text{diam}(\Omega) < a\pi \). Then

\[ V = \{ \exp(i\alpha \cdot x) \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \} \]

is a set of basis functions of \( H \).

Let \( V = \{v_1, v_2, \ldots\} \) be a set of basis functions of \( H \). Let \( \{v_1 \sim, v_2 \sim, \ldots, v_N \sim\} \) be an ortho-normalization of \( \{v_1, v_2, \ldots, v_N\} \) in \( H \), and set

\[
\begin{pmatrix}
    v_1 \\
    \vdots \\
    v_N
\end{pmatrix}
= S_N
\begin{pmatrix}
    v_1 \\
    \vdots \\
    v_N
\end{pmatrix}, \quad S_N = (s_{ij})_{1 \leq i, j \leq N}.
\]

As is well known, we have

Lemma 2.4. For \( w \in H \), set

\[
\begin{align*}
    w^{(N)} &= \sum_{k=1}^{N} [w, v_k \sim] v_k \sim \\
    \end{align*}
\]

Then it holds
\[ \| w^{(N)} - w \| \leq \| \xi - w \| \]
for any \( \xi \in \text{sp}\{v_1, \ldots, v_N\} \).

Lemma 2.5. For \( w \in H \), set
\[
\begin{align*}
W^{(N)} &= \sum_{k=1}^{N} [w, v_k] v_k, \\
K &= \sum_{j=1}^{N} [v_j, v_N] v_N.
\end{align*}
\]
Then it is represented by
\[
W^{(N)} = ( [w, v_1], \ldots, [w, v_N] ) K^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},
\]
where
\[
K = ( [v_j, v_k] )_{j,k=1,\ldots,N}.
\]

Proof. It is clear that
\[
W^{(N)} = ( [w, v_1], \ldots, [w, v_N] ) S_N^* S_N \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}.
\]
On the other hand, we have
\[
[v_1, v_j] = \sum_{p=1}^{N} \sum_{q=1}^{N} s_{p,q} v_p v_q = s_{1,1} v_1 v_1, \quad [v_j, v_N] = s_{j,q} v_j v_q,
\]
that is,
\[
( [v_1, v_j] )_{j=1,\ldots,N} = S_N K S_N^*.
\]
that is,
\[
S_N^* S_N = K^{-1}.
\]

Proposition 2.2. Suppose that \( w \) is a supplementary function of \( H \)-weak solution of \( (P) \). Then it holds
\[
W^{(N)} = ( (f, v_1), \ldots, (f, v_N) ) K^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \rightarrow w \text{ in } H \text{ (as } N \rightarrow \infty).\]

Proof. Let \( w \in H \). Since \( \text{sp } V \) is dense in \( H \), there exists \( w_j \) such that
\[
w_j \in \text{sp}\{v_1, \ldots, v_N(j)\} \text{ (} j \in N(\{N\}) \text{)}, \quad \| w_j - w \| < 1/j
\]
for any positive integer \( j \). Set
\[
W^{(N)} = ( [w, v_1], \ldots, [w, v_N] ) K^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},
\]
then we have from Lemma 2.4.
\[ \| w^{(N)} - w \| \leq \| w_j - w \| \quad (\forall N \geq N(j)) . \]

Therefore we have
\[ \| w^{(N)} - w \| \to 0 \quad (N \to \infty). \]

On the other hand, since \( w \) is a supplementary function of \( H \)-weak solution of \( (P) \), we have
\[ [w, v] = (f, v) \quad (\forall v \in H). \]

Therefore
\[ [w, v_k] = (f, v_k) \quad (k = 1, 2, \ldots). \]

Hence we have
\[ w^{(N)} = ((f, v_1), \ldots, (f, v_N)) K_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}. \]

Let \( w^{(N)} \) be the one defined in Proposition 2.2 and set \( u^{(N)} = A^*w^{(N)} \), then we have
\[ \| u^{(N)} - u \| = \| A^*w^{(N)} - A^*w \| \leq \| w^{(N)} - w \| \to 0 \quad (\text{as } N \to \infty) \]
from the definition of the norm of \( H \). Hence we have

**Theorem III.** Suppose that \( u \) is a \( H \)-weak solution of \( (P) \). Set
\[ u^{(N)} = ((f, v_1), \ldots, (f, v_N)) K_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix}, \]

where \( K_N = (v_{i,j}, v_{j,i})_{1 \leq i, j \leq N} \). Then it holds that
\[ \| u^{(N)} - u \| \to 0 \quad (\text{as } N \to \infty). \]

§ 3. Variational problems.

Let us consider of a variational problem, relating to a continuous anti-linear functional \( \mathcal{J}[v] \) in \( H \). Namely, define a quadratic functional in \( H \):
\[ J[v] = \| v \|^2 - 2 \text{Re} \mathcal{J}[v], \]
and the variational problem of \( J[v] \) in \( H \) is to seek a stationary function \( w \in H \) such that
\[ J[w] = \min_{v \in H} J[v]. \]

Since
\[ J[w + v] - J[w] = 2 \text{Re} \{ [w, v] - \mathcal{J}[v] \} + \| v \|^2 , \]
It is equivalent that
\[ J[w + v] - J[w] \geq 0 \quad (\forall v \in H) . \]
and that
\[ [w, v] - \mathcal{J}[v] = 0 \quad (\forall v \in H) . \]

Here we have
Lemma 3.1. \( w \in H \) is a Riesz' function of \( \mathcal{L}[v] \) in \( H \), i.e.
\[
[w,v] = \mathcal{L}[v] \quad (\forall v \in H),
\]
iff \( w \) is a stationary function of the variational problem of \( J[v] \), i.e.
\[
J[w] = \min_{v \in H} J[v].
\]
Moreover,
\[
\|w\|^2 = \left( \sup_{v \in H} \frac{|\mathcal{L}[v]|}{\|v\|} \right)^2 = - \min_{v \in H} J[v].
\]

Set \( H_\ast = \text{sp}(v_1, \ldots, v_\ast) \), then we have

Lemma 3.2. \( w^{(\ast)} \in H_\ast \) is a Riesz' function of \( \mathcal{L}[v] \) in \( H_\ast \), i.e.
\[
[w^{(\ast)},v] = \mathcal{L}[v] \quad (\forall v \in H_\ast),
\]
iff \( w^{(\ast)} \) is a stationary function of the variational problem of \( J[v] \) in \( H_\ast \), i.e.
\[
J[w^{(\ast)}] = \min_{v \in H_\ast} J[v].
\]
Moreover,
\[
\|w^{(\ast)}\|^2 = \left( \sup_{v \in H_\ast} \frac{|\mathcal{L}[v]|}{\|v\|} \right)^2 = - \min_{v \in H_\ast} J[v].
\]

From Lemma 3.1 and Lemma 3.2, we have

Theorem IV. The supplementary function \( w \) of \( H \)-weak solution of the problem \( (P) \) is a stationary function of the variational problem of
\[
J[v] = \|v\|^2 - 2 \text{ Re}(f,v) \quad \text{in} \quad H,
\]
and
\[
w^{(\ast)} = (v_1, \ldots, (f,v_\ast)) K^{-1}_\ast \begin{pmatrix} v_1 \\ \vdots \\ v_\ast \end{pmatrix}
\]
is a stationary function of the variational problem of
\[
J[v] = \|v\|^2 - 2 \text{ Re}(f,v) \quad \text{in} \quad H_\ast.
\]
References