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Bell Domains and Critical Point Parameters for Triply Connected Planar Domains

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1 Introduction

The fundamental problem in the geometric function theory is to find a family of canonical domains. Recently, S. Bell proposed a new family of domains which admit canonically a simple proper holomorphic map to the unit disc $U$. Actually, they are enough.

Theorem 1 ([3]). Every non-degenerate $d$-ply connected planar domain $W$ with $d > 1$ is mapped biholomorphically (or, conformally) onto a domain $W_{a,b}$, defined by

$$W_{a,b} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{d-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex vectors

$$a = (a_1, a_2, \ldots, a_{d-1}), \quad b = (b_1, b_2, \ldots, b_{d-1}) .$$

This theorem can be considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

We call such a domain $W_{a,b}$ as in Theorem 1 a Bell representation of $W$, or a Bell domain. The function $f_{a,b}$ defined by

$$f_{a,b}(z) = z + \sum_{k=1}^{d-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic map from $W_{a,b}$ onto $U$. Set $B_d$ be the set of all vectors $(a, b)$ in $\mathbb{C}^{2d-2}$ such that $W_{a,b}$ is a Bell representation of $d$-ply connected planar domains, and we call $B_d$ the coefficient body of degree $d$. (Cf. [4].)
Now, from a well-known fact on the theory of moduli, we can conclude that \(d\)-ply connected non-degenerate planar domains have real \(3d - 6\) moduli (or Teichmüller) parameters if \(d \geq 3\). First we state this fact more precisely.

**Definition 1.** Let \(d \geq 2\). We call a \(d\)-ply connected non-degenerate planar domain \(W\) equipped with an order of boundary components of \(W\) a \textit{boundary-marked} planar domain of type \(d\).

Two boundary-marked planar domains \(W_1\) and \(W_2\) of type \(d\) are \textit{conformally equivalent} if there is a conformal map \(f : W_1 \rightarrow W_2\) which preserves the boundary-marking (i.e. the prescribed order of the boundary components).

Let \(D_d\) be the set of all equivalence classes of boundary-marked planar domains of type \(d\). We call \(D_d\) the \textit{deformation space} of a boundary-marked planar domain of type \(d\).

Then the following fact is classical.

**Proposition 2.** If \(d \geq 3\), then \(D_d\) can be considered as a domain in \(\mathbb{R}^{3d-6}\).

**Proof.** By Koebe’s theorem ([5]), every \(d\)-ply connected non-degenerate planar domain can be mapped conformally onto a Koebe circle domain.

On the other hand, it is easy to see that boundary-marked Koebe circle domains have real \(3d - 6\) real global parameters up to Möbius transformations. 

\[\Box\]

## 2 Main results

In this section, we mainly use the symbolic and algebraic computation system Risa/Asir to obtain the defining equations.

In the case of triply connected planar domains, there always exists a canonical symmetry for every such one. As a model of the deformation space of a triply connected planar domain, the intersection of the coefficient body \(B_3\) with one of the real three-dimensional families as follows has been discussed.

**Definition 2.** Set

\[
B^+ = \{(a, b, d) \in \mathbb{R}^3 : a > 0, \ b > 0, \ d > 0\} ,
\]

and

\[
B^- = \{(a, b, d) \in \mathbb{R}^3 : a > 0, \ b < 0, \ d < 0\} .
\]
We assume that $B^\pm$ are naturally embedded in $\mathbb{C}^3$. Also in the sequel, we write as
\[
W_{a,b,d} = \{z \in \mathbb{C} : |f_{a,b,d}(z)| < 1\},
\]
where
\[
f_{a,b,d}(z) = z + \frac{b}{z-a} + \frac{d}{z+a}.
\]

We will discuss about these families, and we show that the critical point parameters on $B^-$ actually give a system of coordinates for $D_3$, which is one of the main theorems.

**Theorem 3.** In the case of $B^-$, the set of three real parameters
\[
(r, s, t)
\]
defined in Lemma 4 below gives the global coordinate system of $B^-$. In other words, the map $\Pi^-$ of $B^-$ to $(r, s, t) \in \mathbb{R}^3$ is a homeomorphism onto the image, where $(0, s, t)$ is identified with $(0, t, s)$.

To show this theorem, first we note the following two lemmas.

**Lemma 4.** For every $f = f_{a,b,d}$ with $(a, b, d) \in B^-$, $f$ has two pair of complex conjugates $\{r+it, r-it\}$ and $\{p+si, p-si\}$ as critical points. Here we assume that
\[
r \leq p, \ t > 0, \ s > 0.
\]

The critical points of $f$ is given by the solutions of
\[
F(z) = (z-a)^2(z+a)^2 - b(z+a)^2 - d(z-a)^2,
\]
and $F(z)$ is represented also as
\[
F(z) = z^4 + \sigma_1 z^3 + \sigma_2 z^2 + \sigma_3 z + \sigma_4.
\]
Clearly, $\sigma_1 = 0$ and the vectors $(\sigma_2, \sigma_3, \sigma_4)$ correspond to the sets $\{r, s, t\}$ bijectively, which is called the relations between solutions and coefficients. Also a direct computation gives the following lemma.

**Lemma 5.** The Jacobian
\[
\frac{\partial(\sigma_2, \sigma_3, \sigma_4)}{\partial( a, b, d)}
\]
is
\[
-8a^2(4a^2 - b - d).
\]
**Proof of Theorem 3.** First, the map

\[ \phi : (a, b, d) \mapsto (\sigma_2, \sigma_3, \sigma_4) \]

is locally homeomorphic by Lemma 5 and the assumptions that \( b < 0 \) and \( d < 0 \). Also \( \phi \) is injective. Indeed, \( a^2 \) is a positive solution of

\[ 3x^2 + \sigma_2 x - \sigma_4 = 0 . \]

And since \( \sigma_4 > 0 \), it has exactly one positive solution (see also Remark 1).

Next, we can show by a direct computation that the Jacobian

\[
\frac{\partial (\sigma_2, \sigma_3, \sigma_4)}{\partial (r, s, t)} = 4st \left( 2(t^2 - s^2)^2 + 16r^2(2r^2 + s^2 + t^2) \right) = 8st \left( 4r^2 + (s - t)^2 \right) \left( 4r^2 + (s + t)^2 \right)
\]

is non-negative, and equals 0 if and only if \( r = 0 \) and \( s = t \). Hence we conclude that

\[ \psi : (r, s, t) \mapsto (\sigma_2, \sigma_3, \sigma_4) \]

is also locally homeomorphic.

Thus we see that the map \( \Pi^- \) of \( B^- \) to \( (r, s, t) \in \mathbb{R}^3 \) is injective and locally homeomorphic, and hence is a homeomorphism onto the image.

**Remark 1.** The image of \( B^- \) by \( \phi \) is

\[
I^- = \left\{ \sigma_2 > 0, \; \sigma_4 > 0, \left|\sigma_3\right| < \frac{\sqrt{6}}{9} \left( 2\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4} \right) \sqrt{-\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4}} \right\}
\]

\[
\cup \left\{ \sigma_2 < 0, \; \sigma_4 > \frac{\sigma_2^2}{4}, \left|\sigma_3\right| < \frac{\sqrt{6}}{9} \left( 2\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4} \right) \sqrt{-\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4}} \right\},
\]

and the map \( \phi : B^- \rightarrow I^- \) is bijective.

On the contrary, the situation is much complicated in case of \( B^+ \).
Theorem 6. In the case $B^+$, the map $\Pi^+ : (a, b, d) \mapsto (r, s, t)$ is locally homeomorphic except for the degenerate locus

$$E_1 = \{(a, b, d) : 4a^2 - b - d = 0\}.$$

The bifurcation locus is

$$E_2 = \{(a, b, d) : (4a^2 - b - d)^3 - 108bda^2 = 0\}.$$

To show this theorem, we note the following lemmas.

Lemma 7. For every $f = f_{a,b,d}$ with $(a, b, d) \in B^+$, either

1) $f$ has four real critical points $\{r, p, s, t\}$, or
2) $f$ has two real critical points $\{r, t\}$ and two others $\{p + si, p - si\}$. Here we may assume that

1) $r < p \leq s < t$, or
2) $r < t$, $s > 0$,

respectively.

Lemma 8. The phase transition occurs at the locus $\text{Discr}(F) = 0$, where

$$F(z) = (z - a)^2(z + a)^2 - b(z + a)^2 - d(z - a)^2.$$

Also, $\text{Discr}(F)$ is

$$64bda^2 \left( (4a^2 - b - d)^3 - 108bda^2 \right).$$

Proof of Theorem 6. The first assertion follows from Lemma 5. And the second assertion is stated in Lemma 8. \qed

Remark 2. On the subset of $B^+$ where $a^2 - b - d > 0$, $\Pi^+$ is injective.

Remark 3. The image $I^+ = \phi(B^+)$ is

$$I^+ = \left\{ \sigma_2 < 0, \ -\frac{\sigma_2^2}{12} < \sigma_4 < \frac{\sigma_2^2}{4}, \right.$$ \[
|\sigma_3| < \frac{\sqrt{6}}{9} \left( -2\sigma_2 - \sqrt{\sigma_2^2 + 12\sigma_4} \right) \sqrt{-\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4}} \right\}. \]

Moreover, $\#\phi^{-1}(\sigma_2, \sigma_3, \sigma_4) = 2$ if and only if $(\sigma_2, \sigma_3, \sigma_4) \in \tilde{I}^+$, where

$$\tilde{I}^+ = \left\{ \sigma_2 < 0, \ -\frac{\sigma_2^2}{12} < \sigma_4 < 0, \right.$$ \[
|\sigma_3| < \frac{\sqrt{6}}{9} \left( -2\sigma_2 + \sqrt{\sigma_2^2 + 12\sigma_4} \right) \sqrt{-\sigma_2 - \sqrt{\sigma_2^2 + 12\sigma_4}} \right\}, \]

and $\#\phi^{-1}(\sigma_2, \sigma_3, \sigma_4) = 1$ if and only if $(\sigma_2, \sigma_3, \sigma_4) \in I^+ - \tilde{I}^+$.
Next, we give the explicit description of the boundaries of $B^\pm \cap B_3$ in $B^\pm$. Here, recall the following proposition.

**Proposition 9** ([4]). $W_{a,b}$ belongs to $B_d$ if and only if all of critical values belongs to the open unit disc $U$.

Thus we can see the following corollary.

**Corollary 1.** The boundary of $B^- \cap B_3$ in $B^-$ is contained in the coordinate planes

$$
\{(a, b, d) : a = 0\}, \quad \{(a, b, d) : b = 0\}, \quad \{(a, b, d) : d = 0\},
$$

and the loci

$$
\{(a, b, d) : |f_{a,b,d}(r + it)| = 1\}, \quad \{(a, b, d) : |f_{a,b,d}(-r + is)| = 1\} .
$$

**Corollary 2.** The boundary of $B^+ \cap B_3$ in $B^+$ is contained in the coordinate planes

$$
\{(a, b, d) : a = 0\}, \quad \{(a, b, d) : b = 0\}, \quad \{(a, b, d) : d = 0\},
$$

the loci

Result$(F(z), G(z) - (z^2 - a^2)) = 0$, \quad Result$(F(z), G(z) + (z^2 - a^2)) = 0$

and

$$
\{(a, b, d) : |f_{a,b,d}(-(r + t)/2 + i s)| = 1\} ,
$$

where

$$
G(z) = z(z - a)(z + a) + b(z + a) + d(z - a) .
$$

On the other hand, if we set $z = x + iy$, the condition $|R(z)| = 1$ implies that

$$
x^6 + (3y^2 - 2a^2 + 2b + 2d - 1)x^4 + (2b - 2d)ax^3 + (3y^4 - 2y^2 + a^4

+ (-2b - 2d + 2)a^2 + b^2 + 2db + d^2)x^2 + ((-6b + 6d)ay^2 + (-2b + 2d)a^3

+ (2b^2 - 2d^2)a)x + y^6 + (2a^2 - 2b - 2d - 1)y^4 + (a^4 + (-2b - 2d - 2)a^2

+ b^2 + 2db + d^2)y^2 - a^4 + (b^2 - 2db + d^2)a^2 = 0 . \quad (1)
$$

Furthermore, since $F(z) = 0$, we have

$$
x^4 - (6y^2 + 2a^2 + b + d)x^2 - (2b - 2d)ax + y^4 + (2a^2 + b + d)y^2

+ a^4 - (b + d)a^2 = 0 \quad (2)
$$
and
\[ y(2x^3 - (2y^2 + 2a^2 + b + d)x - (b - d)a) = 0. \]

Thus if \( y = 0 \), we have the boundary loci \( L_1 \) defined by
\[
4a^6 - (12b + 12d + 8)a^4 + (-36b + 36d)a^3 \\
- (15b^2 - (78d - 20)b + 15d^2 + 20d - 4)a^2 + (-18b^2 + 4b + 18d^2 - 4d)a \\
- 4b^3 - (12d - 1)b^2 - (12d^2 - 2d)b - 4d^3 + d^2 = 0
\]
or \( L_{-1} \) defined by
\[
4a^6 - (12b + 12d + 8)a^4 + (36b - 36d)a^3 \\
- (15b^2 - (78d - 20)b + 15d^2 + 20d - 4)a^2 + (18b^2 - 4b - 18d^2 + 4d)a - 4b^3 \\
- (12d - 1)b^2 - (12d^2 - 2d)b - 4d^3 + d^2 = 0
\]
according as the critical value is 1 or \(-1\), respectively.

Otherwise, computing Gröbner basis for the ideal generated by three polynomials \((1), (2), \) and \(2x^3 - (2y^2 + 2a^2 + b + d)x - (b - d)a = 0\) with block order \([x, y] > [a, b, c]\), and finding polynomial maps belong to \(\mathbb{C}[a, b, d]\), we have the defining equation of boundary locus \(L\).

\[
64a^{18} - 64(9b + 9d - 2)a^{16} + 16(63b^2 + (450d - 28)b + 63d^2 - 28d - 4)a^{14} \\
+ 16(150b^3 - (1494d + 109)b^2 - 2(747d^2 - 107d - 12)b + 150d^3 - 109d^2 + 24d \\
- 16)a^{12} \\
- 4(657b^4 + (684d - 800)b^3 + (-34938d^2 - 1536d + 168)b^2 + (684d^3 + 1536d^2 \\
+ 624d - 32)b + 657d^4 - 800d^3 + 168d^2 - 32d + 16)a^{10} \\
- 4(2097b^5 + (-16083d - 2276)b^4 + (46242d^2 + 10552d - 32)b^3 + (46242d^3 \\
+ 2328d^2 - 1248d + 584)b^2 + (-16083d^4 + 10552d^3 - 1248d^2 - 2352d - 48)b \\
+ 2097d^5 - 2276d^4 - 32d^3 + 584d^2 - 48d - 32)a^8 \\
- (7503b^6 - 2(29493d + 3526)b^5 + (221409d^2 + 33860d - 1740)b^4 - 4(170979d^3 \\
+ 24110d^2 - 1932d - 608)b^3 + (221409d^4 - 96440d^3 + 3384d^2 + 4608d + 336)b^2 \\
+ (-58986d^5 + 33860d^4 + 7728d^3 + 4608d^2 + 96d - 320)b + 7503d^6 - 7052d^5 \\
- 1740d^4 + 2432d^3 + 336d^2 - 320d - 64)a^6 \\
- (3276b^7 - (15948d + 2207)b^6 + (14364d^2 + 798d - 1176)b^5 + (145764d^3 \\
+ 34719d^2 + 5064d + 536)b^4 + 4(36441d^4 + 15857d^3 - 2508d^2 - 304d + 56)b^3 \\
+ (14364d^5 + 34719d^4 - 10032d^3 - 10416d^2 - 480d + 16)b^2 - 2d(7974d^5 
\]
\[-399d^4 - 2532d^3 + 608d^2 + 240d - 16)b + d^2(3276d^5 - 2207d^4 - 1176d^3
+ 536d^2 + 224d + 16))a^4
- 8(b + d)^3(90b^5 - (198d + 25)b^4 - (1044d^2 + 208d + 28)b^3 - (1044d^3 + 366d^2
- 60d + 4)b^2 - (198d^4 + 208d^3 - 60d^2 - 24d)b + 90d^5 - 25d^4 - 28d^3 - 4d^2)a^2
- 16(b + d)^8(4b + 4d + 1) = 0\,.

Thus we conclude the following.

**Theorem 10.** The boundary of $B^\pm \cap B_3$ in $B^\pm$ is contained in the coordinate planes

\[\{(a, b, d) : a = 0\}, \quad \{(a, b, d) : b = 0\}, \quad \{(a, b, d) : d = 0\}\]

and the locus $L$.

The boundary of $B^\pm \cap B_3$ in $B^\pm$ is contained in the coordinate planes

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and the loci $L$, $L_1$ and $L_{-1}$.

**References**


Bell Domains and Critical Point Parameters for Triply Connected Planar Domains

Mohaby Karima

In this note, we consider deformation of a non-degenerate triply connected planar domain. Here, the Bell representation with suitable real parameters is crucial. For two fundamental families of Bell domains, we determine whether the systems of critical point parameters are faithful or not, and the boundary of these families are calculated explicitly.

More precisely, we consider two families:

\[ B^+ = \{(a, b, d) \in \mathbb{R}^3 \mid a > 0, b > 0, d > 0\} \]

and

\[ B^- = \{(a, b, d) \in \mathbb{R}^3 \mid a > 0, b < 0, d < 0\} \]

in the case of \( d = 3 \). And among other things, we show that the set of three real parameters \((r, s, t)\) defined from critical points as in the section 2 gives the set of global coordinates for \( B^- \).