$h$ transform of one dimensional generalized diffusion operators∗

Tomoko Takemura  
Department of Mathematics, Nara Women’s University,  
Kita-Uoya Nishimachi, Nara, 630-8506 Japan  
e-mail: sm18031@cc.nara-wu.ac.jp

Matsuyo Tomisaki  
Department of Mathematics, Nara Women’s University,  
Kita-Uoya Nishimachi, Nara, 630-8506 Japan  
e-mail: tomisaki@cc.nara-wu.ac.jp

Abstract

We are concerned with two types of $h$ transform of one dimensional generalized diffusion operators treated by Maeno(2006) and by the second author(2007). We show that these two types of $h$ transform are in inverse relation to each other in some sense. Further we show that a recurrent one dimensional generalized diffusion operator cannot be represented as an $h$ transform of another one dimensional generalized diffusion operator different from the original one. We also consider a spectral representation of elementary solutions corresponding to $h$ transformed one dimensional generalized diffusion operators.

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1 Introduction

Let \( G_{s,m,k} \) be a one dimensional generalized diffusion operator on an interval \( I \) with scale function \( s \), speed measure \( m \), and killing measure \( k \) (ODGDO with \((s,m,k)\) for brief). Let \( \mathcal{M}_{s,m} \) be the set of all positive functions \( h \) satisfying \( G_{s,m,0}h \leq 0 \), where \( G_{s,m,0} \) is an ODGDO with \((s,m,0)\) and 0 denotes the null killing measure. For \( \beta \geq 0 \), let \( \mathcal{H}_{s,m,k,\beta} \) be the set of all positive functions satisfying \( G_{s,m,k}h = \beta h \). In this paper, we are concerned with two types of \( h \) transform based on \( h \in \mathcal{M}_{s,m} \) and \( h \in \mathcal{H}_{s,m,k,\beta} \).

It is well known that the generator of one dimensional generalized diffusion process (ODGDP for brief) is represented as an ODGDO. In Section 4.3 of [3] K. Itô and H. P. McKean used a method to derive such general form \( G_{s,m,k} \) for the generator of ODGDP. Their method is an \( h \) transform based on the probability which the sample paths hit one of the end points of the state interval. Following their idea, Maeno [4] considered \( h \) transforms of ODGDPs based on \( h \in \mathcal{M}_{s,m} \) and studied some properties corresponding to \( h \) transformed ODGDP. On the other hand, the second author [8] considered \( h \) transforms of ODGDOs based on \( h \in \mathcal{H}_{s,m,k,\beta} \) and obtained some results for \( h \) transformed ODGDOs. As we will see in Proposition 1.1 below, the sets \( \mathcal{M}_{s,m} \) and \( \mathcal{H}_{s,m,k,\beta} \) are usually disjoint each other. Therefore the results of [4] and [8] are not necessarily derived from each other.

Proposition 1.1 (i) The set \( \mathcal{H}_{s,m,0,0} \) coincides with the set \( \{h \in \mathcal{M}_{s,m}; D_s h(x) \text{ is a constant function on } I\} \).
(ii) If \( k \) is not a null measure or \( \beta > 0 \), then \( \mathcal{M}_{s,m} \cap \mathcal{H}_{s,m,k,\beta} = \emptyset \).

Here \( D_s h \) is the right derivative of \( h \) with respect to the scale function \( s \). We show this proposition in Section 3.

Let \( p(t,x,y) \) be the elementary solution of the equation
\[
\frac{\partial}{\partial t} p(t,x,y) = G_{s,m,k} p(t,x,y), \quad t > 0, \ x, y \in I,
\] (1.1)
in the sense of McKean [5], where \( G_{s,m,k} \) is applied to \( x \) or \( y \).

Let \( p'(t,x,y) \) be the elementary solution of the equation (1.1) with \( G_{s,m,k} \).
replaced by $\mathcal{G}_{s,m,0}$. For $h \in \mathcal{M}_{s,m}$, we set

$$p^h_0(t,x,y) = p^o(t,x,y)/h(x)h(y), \quad t > 0, \ x, y \in I,$$

(1.2)

$$s_h(x) = \int_{(c_o,x]} h(y)^{-2} ds(y),$$

(1.3)

$$m_h(x) = \int_{(c_o,x]} h(y)^2 dm(y),$$

(1.4)

$$k_h(x) = -\int_{(c_o,x]} h(y) dD_h(y),$$

(1.5)

where $c_o$ is a point of $I$ fixed arbitrarily. Maeno [4] showed that $p^h_0(t,x,y)$ is the elementary solution of the equation (1.1) with $\mathcal{G}_{s,m,k,0}$ in place of $\mathcal{G}_{s,m,k}$. Further she studied the asymptotic behavior near the boundaries of $I$ for sample paths of ODGDP with generator $\mathcal{G}_{s,m,k,h}$, and gave a precise classification of the states of the boundaries by means of $s$, $m$ and $h$. We call $\mathcal{G}_{s,m,k,0}$ the $h$ transform of $\mathcal{G}_{s,m,0}$ with $h \in \mathcal{M}_{s,m}$.

We next turn to an $h$ transform treated in [8]. Let $p(t,x,y)$ be the elementary solution of the equation (1.1). For $h \in \mathcal{H}_{s,m,k,0}$ set

$$p^*_h(t,x,y) = e^{-\beta t}p(t,x,y)/h(x)h(y), \quad t > 0, \ x, y \in I.$$  

(1.6)

Then $p^*_h(t,x,y)$ is the elementary solution of the equation (1.1) with $\mathcal{G}_{s,m,k,0}$ in place of $\mathcal{G}_{s,m,k}$, where $s_h$ and $m_h$ are given by (1.3) and (1.4), respectively. We note that the first author [7] studied some asymptotic properties of sample paths near the boundaries of $I$ for ODGDP with generator $\mathcal{G}_{s,m,0}$, and obtained a precise classification of the states of the boundaries by means of $s$, $m$, $k$ and $h$. We call $\mathcal{G}_{s,m,0}$ the $h$ transform of $\mathcal{G}_{s,m,k}$ with $h \in \mathcal{H}_{s,m,k,0}$.

For $h \in \mathcal{M}_{s,m}$ [resp. $h \in \mathcal{H}_{s,m,k,0}$], we set $H^o_h\mathcal{G}_{s,m,0}u = h^{-1}\mathcal{G}_{s,m,0}(hu)$ [resp. $H^*h\mathcal{G}_{s,m,k+\beta m}u = h^{-1}\mathcal{G}_{s,m,k+\beta m}(hu)$] for $u$ satisfying $hu \in D(\mathcal{G}_{s,m,0})$ [resp. $hu \in D(\mathcal{G}_{s,m,k+\beta m})$]. In the case that $\mathcal{G}_{s,m,0}$ and $\mathcal{G}_{s,m,k}$ are differential operators of second order, it is easy to see that $H^o_h\mathcal{G}_{s,m,0}$ and $H^*\mathcal{G}_{s,m,k+\beta m}$ coincide with $\mathcal{G}_{s,m,k,h}$ and $\mathcal{G}_{s,m,k,h}$, respectively. We show that this is also true for ODGDOs.

**Theorem 1.2** Let $u$ be a measurable function on $I$.

(i) Let $h \in \mathcal{M}_{s,m}$. Then $u$ belongs to $D(\mathcal{G}_{s,m,k,h})$ if and only if $hu$ belongs to $D(\mathcal{G}_{s,m,0})$. Further $\mathcal{G}_{s,m,k,h}u = H^o_h\mathcal{G}_{s,m,0}u$ holds true for $u \in D(\mathcal{G}_{s,m,k,h})$. 

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Let $h \in H_{s,m,k,\beta}$. Then $u$ belongs to $D(G_{s,m,0})$ if and only if $hu$ belongs to $D(G_{s,m,k})$. Further $G_{s,m,0}u = H^{*}_{h}G_{s,m,k+\beta m}u$ holds true for $u \in D(G_{s,m,0})$.

**Corollary 1.3** (i) Let $h \in H_{s,m,0,0}$. Then $H^{o}_{h}G_{s,m,0} = H^{o}_{h}G_{s,m,0} = G_{s,m,0}$.

(ii) Let $h \in M_{s,m}$. Then $H^{o}_{h}G_{s,m,0} = G_{s,m,0}$ if and only if $h$ is a positive constant function.

(iii) Let $h \in H_{s,m,k,\beta}$. Then $H^{o}_{h}G_{s,m,k+\beta m} = G_{s,m,k+\beta m}$ if and only if $k$ is the null measure, $\beta = 0$, and $h$ is a positive constant function.

We next show that $H^{o}_{h}$ and $H^{*}_{h}$ are in inverse relation to each other in some sense.

**Theorem 1.4** (i) Let $h \in M_{s,m}$. Then $h^{-1}$ belongs to $H_{s,m,k+\beta m}$ and $H^{-1}_{h}H^{o}_{h}G_{s,m,0} = G_{s,m,0}$.

(ii) Let $h \in H_{s,m,k,\beta}$. Then $h^{-1}$ belongs to $M_{s,m}$, and $H^{-1}_{h}H^{o}_{h}G_{s,m,k+\beta m} = G_{s,m,k+\beta m}$.

We should note that Theorem 1.4 does not necessarily ensure one-to-one correspondence of $h$ transform. Indeed, some examples are given in Section 6, which show that $h$ transform of ODGDO is not one-to-one correspondence.

Now Theorem 1.2 implies that an ODGDO is represented as an $h$ transform of another ODGDO, but it does not always imply that an ODGDO is represented as an $h$ transform of another ODGDO different from the original one. As we will see in the following theorem, this problem is related to a global property of ODGDO. We denote by $\Phi$ the set of all ODGDOs on $I$. Let $\Phi^{R}$ be the set of all recurrent ODGDOs on $I$, that is,

$$\Phi^{R} = \{ G_{s,m,k} \in \Phi : s(l_1) = -\infty, \ s(l_2) = \infty, \text{ and } k \text{ is the null measure} \}.$$  

We set $\Phi^{T} = \Phi \setminus \Phi^{R}$, which is the set of all transient ODGDOs on $I$.

**Theorem 1.5** (i) Let $G_{s,m,0} \in \Phi^{R}$. If $H^{o}_{h}G_{s,m,0} \in \Phi^{R}$ for some $h \in M_{s,m}$, then $h$ is a positive constant function. If $H^{*}_{h}G_{s,m,0} \in \Phi^{R}$ for some $h \in H_{s,m,0,\beta}$, then $h$ is a positive constant function and $\beta = 0$.

(ii) Let $G_{s,m,0} \in \Phi^{R}$. If $H^{o}_{h}G_{s,m,0} \in \Phi^{T}$ for some $h \in M_{s,m}$, then $h$ is not a positive constant function. If $H^{*}_{h}G_{s,m,0} \in \Phi^{T}$ for some $h \in H_{s,m,0,\beta}$, then $h$ is not a positive constant function or $\beta > 0$. Conversely, if $h \in M_{s,m}$ and
If $h$ is not a positive constant function, then $H_h^0 G_{s,m,0} \in \Phi^T$. If $h \in H_{s,m,0,\beta}$ and either $h$ is not a positive constant function or $\beta > 0$, then $H_h^0 G_{s,m,\beta m} \in \Phi^T$.

(iii) If $G_{s,m,0} \in \Phi^T$, then $H_h^0 G_{s,m,0} \in \Phi^T$ for any $h \in \mathcal{M}_{s,m}$. If $G_{s,m,k} \in \Phi^T$, then $H_h^0 G_{s,m,k+\beta m} \in \Phi^T$ for any $h \in H_{s,m,k,\beta}$.

We will show this theorem in Section 4.

We finally consider spectral representations of elementary solutions. We assume that $\text{supp}[m] = I$ and $l_1$ is not $(s, m, k)$-natural. Further we assume that $p(t, x, y)$ is represented as

$$p(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda), \quad t > 0, \ x, y \in I, \quad (1.7)$$

where $d\sigma(\lambda)$ is a Borel measure on $[0, \infty)$ such that $\sigma(\{0\}) = 0$ if $l_1$ is $(s, m, k)$-regular or -exit. Further $\psi(x, \lambda), x \in I, \lambda \geq 0$, is the solution of the integral equation (5.1) or (5.2) below. Let $h \in H_{s,m,k,\beta}$ and $p_h^*(t, x, y)$ be the elementary solution of the equation (1.1) with $G_{s_h,m_h,0}$ in place of $G_{s,m,k}$, which is given by (1.6). By means of (1.6) and (1.7), we suppose $p_h^*(t, x, y)$ is represented as

$$p_h^*(t, x, y) = \int_{[\beta, \infty)} e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \quad t > 0, \ x, y \in I, \quad (1.8)$$

and $\psi_h^*(x, \lambda)$ and $d\sigma_h^*(\lambda)$ are represented as

$$\psi_h^*(x, \lambda) = C_0 \psi(x, \lambda - \beta)/h(x), \quad d\sigma_h^*(\lambda) = C_0^{-2} d\lambda \sigma(\lambda - \beta), \quad (1.9)$$

for $\lambda \geq \beta$, where $C_0$ is a positive constant function. In Section 5 we show that $\psi_h^*(x, \lambda)$ satisfies (5.1) or (5.2) with $(s, m, k)$ replaced by $(s_h, m_h, k)$. Further we will find there that $C_0$ only depends on behavior of $s, m, k$ and $h$ near the boundary $l_1$, which leads us to an interesting behavior of Lévy measure density of inverse local time corresponding to a diffusion process on $I$ with the scale function $s_h$ and the speed measure $m_h$. We will discuss it in another paper.

The organization of this paper is as follows. In Section 2 we give the precise definitions of ODGD $G_{s,m,k}$, the domain $D(G_{s,m,k})$ and the corresponding items. In Section 3 we give the precise definitions of $\mathcal{M}_{s,m}$ and $H_{s,m,k,\beta}$, and prove Proposition 1.1 and Theorems 1.2, 1.4. In Section 4 we prove Theorem 1.5. In Section 5 we show that (1.8) holds true for $\psi_h^*(x, \lambda)$.
and $d\sigma^*_x(\lambda)$ given by (1.9) with a suitable constant $C_o$. It is easy to see that if $H^o_h G^s_{m,0} = H^o_h G^s_{m,0} \ldots$ for some $h_1, h_2 \in \mathcal{M}_{s,m}$ [resp. $h_1, h_2 \in H_{s,m,k,+\beta}$], then there is a positive constant $K$ such that $h_1 = Kh_2$. However it is not necessarily true that if $H^o_h G^s_{m,1,0} = H^o_h G^s_{m,2,0}$ [resp. $H^o_h G^s_{m,1,1} = H^o_h G^s_{m,2,2}$] for some $h_1 \in \mathcal{M}_{s,m}$ [resp. $h_1 \in H_{s,m,k,\beta}$] $(i = 1, 2)$, then $h_1 = Kh_2$ for some positive constant $K$ and $G^s_{1,1,0} = G^s_{2,2,0}$ [resp. $G^s_{1,1,1} = G^s_{2,2,2}$]. We give such typical examples in Section 6.

2 Preliminaries

In this section we give the precise definitions of ODGDO $G^s_{m,k}$, the domain $D(G^s_{m,k})$ and the corresponding items.

Let $s$ be a continuous increasing function on an open interval $I = (l_1, l_2)$, where $-\infty \leq l_1 < l_2 \leq \infty$, $m$ be a right continuous nondecreasing function on $I$ and $k$ be a right continuous nondecreasing function on $I$. We sometimes use the same symbols $s$, $m$ and $k$ for the induced measures $d\sigma(x)$, $dm(x)$ and $dk(x)$, respectively. For a function $u$ on $I$, we set $u(l_i) = \lim_{x \to l_i, x \in I} u(x)$ if there exists the limit, for $i = 1, 2$. We set

$$I_u(\mu) = \{x \in I; \mu(x_1) < \mu(x_2) \text{ for } \ell_1 < x_1 < x < x_2 < \ell_2\}, \quad (2.1)$$

for a nondecreasing right continuous function $\mu$ on $I$. $I_u(\mu)$ is the same as the support of the measure induced by $\mu$. We assume $I_u(m) \neq \emptyset$ and $I_u(k) \subset I_u(m)$ throughout this paper. Further we set

$$I_u(m) = I_u(m) \cup \{x; x = \ell_i \text{ with } |m(\ell_i)| + |s(\ell_i)| + |k(\ell_i)| < \infty, \quad i = 1, 2\}.$$

Let us fix a point $c_o \in I_u(m)$ arbitrarily and set

$$J_{\mu,\nu}(x) = \int_{(c_o,x]} d\mu(y) \int_{(c_o,y]} d\nu(z),$$

for $x \in I$, where $\mu$ and $\nu$ are Borel measures on $I$, and the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a > b$. Following [1], we call the boundary $l_i$ to be

- $(s, m, k)$-regular if $J_{s,m+k}(l_i) < \infty$ and $J_{m+k,s}(l_i) < \infty$,
- $(s, m, k)$-exit if $J_{s,m+k}(l_i) < \infty$ and $J_{m+k,s}(l_i) = \infty$,
- $(s, m, k)$-entrance if $J_{s,m+k}(l_i) = \infty$ and $J_{m+k,s}(l_i) < \infty$,
- $(s, m, k)$-natural if $J_{s,m+k}(l_i) = \infty$ and $J_{m+k,s}(l_i) = \infty$. 

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Let $D(G_{s,m,k})$ be the space of all functions $u \in L^2(I, m)$ which have a continuous version $u$ (we use the same symbol) satisfying the following conditions:

i) There exist two constants $A$, $B$ and a function $f_u \in L^2(I, m)$ such that

$$u(x) = A + Bs(x) + \int_{(c_n, x]} \{s(x) - s(y)\} f_u(y) \, dm(y)$$

$$+ \int_{(c_n, x]} \{s(x) - s(y)\} u(y) \, dk(y), \quad x \in I. \tag{2.2}$$

ii) If $l_i$ is $(s, m, k)$-regular, then $u(l_i) = 0$ for each $i = 1, 2$.

By virtue of (2.2), $f_u$ is uniquely determined as a function of $L^2(I, m)$ if it exists. The operator $G_{s,m,k}$ from $D(G_{s,m,k})$ into $L^2(I, m)$ is defined by $G_{s,m,k}u = f_u$, and it is called the one-dimensional generalized diffusion operator with the scale function $s$, the speed measure $m$, and the killing measure $k$ (ODGDO with $(s, m, k)$ for brief). The above condition ii) implies that the absorbing boundary condition is posed at the regular boundary.

It is easy to see that $G_{s_1,m_1,k_1}$ coincides with $G_{s_2,m_2,k_2}$ if and only if there are a positive constant $K$ and constants $K_i$, $i = 1, 2, 3$ such that $s_1 = Ks_2 + K_1$, $m_1 = K^{-1}m_2 + K_2$, and $k_1 = K^{-1}k_2 + K_3$ (see [3]).

We note that $l_i$ is $(s, m, k + \beta m)$-regular, exit, entrance, or natural according to $l_i$ is $(s, m, k)$-regular, exit, entrance, or natural, for $i = 1, 2$ and $\beta \geq 0$. Combining this fact with (2.2), we immediately obtain the following.

$$D(G_{s,m,k}) = D(G_{s,m,k+\beta m}), \tag{2.3}$$

$$G_{s,m,k}u - \beta u = G_{s,m,k+\beta m}u \quad \text{for} \quad u \in D(G_{s,m,k}), \tag{2.4}$$

where $\beta \geq 0$.

In the following, for a measurable functions $u$ on $I$, $D_s u(x)$ stands for the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{\varepsilon \to 0} \{u(x + \varepsilon) - u(x)\}/\{s(x + \varepsilon) - s(x)\}$, provided it exists. It is obvious that $u \in D(G_{s,m,k})$ has the right derivative $D_s u$ and it satisfies

$$D_s u(y) - D_s u(x) = \int_{(x,y]} G_{s,m,k} u(z) \, dm(z) + \int_{(x,y]} u(z) \, dk(z), \quad x, y \in I.$$ 

So we sometimes use the symbol $G_{s,m,k} u = (dD_s u - u \, dk)/dm$.

Following McKean [5] (see also Section 4.11 of [3]), we can define the elementary solution $p(t, x, y)$ of the equation (1.1). It is known that $p(t, x, y)$
satisfies the following properties.

\[
p(t, x, y) = p(t, y, x) > 0, \quad t > 0, \, x, y \in I.
\]
\[
p(t, x, y) \text{ is continuous on } (0, \infty) \times I \times I.
\]
\[
p(s + t, x, y) = \int_I p(s, x, z)p(t, z, y) \, dm(z), \quad s, t > 0, \, x, y \in I.
\]
\[
p(t, l_i, y) = 0, \quad t > 0, \, y \in I, \quad \text{if } l_i \text{ is not entrance}.
\]
\[
D_s p(t, l_i, y) = 0, \quad t > 0, \, y \in I, \quad \text{if } l_i \text{ is entrance},
\]

where \( D_s p(t, x, y) = \lim_{\varepsilon \downarrow 0} \{ p(t, x + \varepsilon, y) - p(t, x, y) \} / \{ s(x + \varepsilon) - s(x) \} \). It is also known that there exists a one-dimensional generalized diffusion process (ODGDP for brief) \( D_{s,m,k} = \{ X(t) : t \geq 0, \, P_x : x \in I^{**}(m) \} \) such that

\[
P_x(X(t) \in E) = \int_E p(t, x, y) \, dm(y), \quad t > 0, \, x \in I^{**}(m), \, E \in \mathcal{B}(I),
\]

where \( \mathcal{B}(I) \) stands for the set of all Borel sets of \( I \). By this reason, \( p(t, x, y) \) is sometimes called the transition probability density with respect to \( m \).

For \( \alpha \geq 0 \) and \( i = 1, 2 \), let \( g_i(\cdot, \alpha) \) be a function on \( I \) satisfying the following properties (2.5)–(2.9).

\[
g_i(x, \alpha) \text{ is positive and continuous in } x. \tag{2.5}
\]
\[
g_1(x, \alpha) \text{ is nondecreasing in } x. \tag{2.6}
\]
\[
g_2(x, \alpha) \text{ is nonincreasing in } x. \tag{2.7}
\]
\[
g_i(l_i, \alpha) = 0 \text{ if } |s(l_i)| < \infty. \tag{2.8}
\]
\[
g_i(x, \alpha) = g_i(c_o, \alpha) + D_s g_i(c_o, \alpha)\{s(x) - s(c_o)\}
\]
\[
+ \int_{(c_o,x]} \{s(x) - s(y)\} g_i(y, \alpha)\{\alpha \, dm(y) + dk(y)\}, \quad x \in I. \tag{2.9}
\]

Here \( D_s g_i(x, \alpha) = \lim_{\varepsilon \downarrow 0} \{ g_i(x + \varepsilon, \alpha) - g_i(x, \alpha) \} / \{ s(x + \varepsilon) - s(x) \} \), \( i = 1, 2 \). It is known that there exist functions \( g_i(\cdot, \alpha), \quad i = 1, 2 \), satisfying the properties (2.5)–(2.9) (see Section 4.6 of [3]). We summarize some properties of \( g_i(\cdot, \alpha), \quad i = 1, 2 \), which we need later.
Proposition 2.1 (i) ([3]) Assume that $k$ is not a null measure or $\alpha > 0$. Then it holds true that

$$g_i(l_i, \alpha) \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-entrance}, \\ = 0 & \text{if } l_i \text{ is not } (s, m, k)\text{-entrance}; \end{cases}$$

$$g_j(l_i, \alpha) \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or exit}, \\ = \infty & \text{if } l_i \text{ is } (s, m, k)\text{-entrance or natural}; \end{cases}$$

$$|D_s g_i(l_i, \alpha)| \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or exit}, \\ = 0 & \text{if } l_i \text{ is } (s, m, k)\text{-entrance or natural}; \end{cases}$$

$$|D_s g_j(l_i, \alpha)| \begin{cases} \in (0, \infty) & \text{if } l_i \text{ is } (s, m, k)\text{-regular or entrance}, \\ = \infty & \text{if } l_i \text{ is } (s, m, k)\text{-exit or natural}; \end{cases}$$

$$\lim_{x \to l_i} g_i(x, \alpha) D_s g_j(x, \alpha) = 0 \quad \text{if } l_i \text{ is } (s, m, k)\text{-exit},$$

$$\lim_{x \to l_i} g_j(x, \alpha) D_s g_i(x, \alpha) = 0 \quad \text{if } l_i \text{ is } (s, m, k)\text{-entrance};$$

where $i, j = 1, 2$ and $i \neq j$.

(ii) Assume that $k$ is the null measure and $\alpha = 0$. Then $g_i(x, 0), i = 1, 2$ are represented as follows.

$$g_1(x, 0) = \begin{cases} C_1 & \text{if } s(l_1) = -\infty \\ C_1 \{s(x) - s(l_1)\} & \text{if } s(l_1) > -\infty, \end{cases}$$

$$g_2(x, 0) = \begin{cases} C_2 & \text{if } s(l_2) = \infty \\ C_2 \{s(l_2) - s(x)\} & \text{if } s(l_2) < \infty, \end{cases}$$

where $C_1$ and $C_2$ are positive constants.

The statement (i) is shown in Section 4.6 of [3]. The statement (ii) follows from (2.5) – (2.8). So we omit the proof.

We set $W(\alpha) = D_s g_1(x, \alpha) g_2(x, \alpha) - g_1(x, \alpha) D_s g_2(x, \alpha)$. Note that $W(\alpha)$ is a positive number independent of $x \in I$. We put

$$G(\alpha, x, y) = G(\alpha, y, x) = W(\alpha)^{-1} g_1(x, \alpha) g_2(y, \alpha), \quad (2.10)$$

for $\alpha > 0, x, y \in I, x \leq y$. We call $G(\alpha, x, y)$ the $\alpha$-Green function corresponding to the ODGDO $G_{s, m, k}$. It is also known that

$$G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) \, dt, \quad \alpha > 0, x, y \in I_*(m). \quad (2.11)$$
It is easy to see that, if \( k \neq 0 \), then there exists \( G(0, x, y) \) which is given by
\[
G(0, x, y) = G(0, y, x) = W^{-1} g_1(x) g_2(y), \quad x, y \in I, \ x \leq y,
\]
(2.12)
where \( g_i(x) = g_i(x, 0), \ i = 1, 2, \) and \( W = D_a g_1(x) g_2(x) - g_1(x) D_a g_2(x) \), which is a positive constant independent of \( x \in I \). It follows from Proposition 2.1 that, in the case \( k = 0 \), there exists \( G(0, x, y) \in (0, \infty) \) if and only if \( |s(l)| < \infty \) for \( i = 1 \) or \( 2 \).

We denote by \( G_\alpha (\alpha > 0) \) the Green operator corresponding to \( G_{s,m,k} \).
\[
G_\alpha f(x) = \int_I G(\alpha, x, y) f(y) \ dm(y), \quad f \in L^2(I, m).
\]
(2.13)
It is well known that
\[
G_\alpha (L^2(I, m)) = D(G_{s,m,k}),
\]
(2.14)
\[
G_\alpha (\alpha - G_{s,m,k}) u = u, \quad u \in D(G_{s,m,k}),
\]
(2.15)
\[
(\alpha - G_{s,m,k}) G_\alpha f = f, \quad f \in L^2(I, m),
\]
(2.16)
(see [2] and [3]).

3 \( h \) transform of ODGDOs

In this section, we give the precise definitions of \( \mathcal{M}_{s,m} \) and \( \mathcal{H}_{s,m,k,\beta} \), and prove Proposition 1.1 and Theorems 1.2, 1.4. We use the same notations as in the preceding sections.

3.1 \( h \) transform of \( G_{s,m,0} \) with \( h \in \mathcal{M}_{s,m} \)

Let \( \mathcal{M}_{s,m} \) be the set of all positive continuous functions \( h \) on \( I \) such that \( h \) has the right derivative \( D_r h \), \( D_s h \) is right continuous and nonincreasing, and the set \( \{ x \in I; D_s h(x_1) > D_s h(x_2) \} \) for \( l_1 < x_1 < x < x_2 < l_2 \) is included in \( I_s(m) \). For \( h \in \mathcal{M}_{s,m} \), we consider \( s_h, m_h \) and \( k_h \) given by (1.3), (1.4) and (1.5), respectively. Note that \( I_s(m) = I_s(m_h) \) and \( I_s(k_h) \subset I_s(m_h) \) for \( h \in \mathcal{M}_{s,m} \) (see [4]). Let \( p^\alpha(t, x, y) \) be the elementary solution of (1.1) with \( G_{s,m,k} \) replaced by \( G_{s,m,0} \). Let \( G_\alpha^\alpha(\alpha, x, y) \) be the \( \alpha \)-Green function corresponding to the ODGDO \( G_{s,m,0} \). For \( h \in \mathcal{M}_{s,m} \), we consider \( p_h^\alpha(t, x, y) \) given by (1.2) and set
\[
G^\alpha_h(\alpha, x, y) = G_\alpha^\alpha(\alpha, x, y)/h(x)h(y), \quad x, y \in I.
\]
(3.1)
The following result is obtained by Maeno (see Theorem 2.2 of [4]).

**Proposition 3.1** ([4]) \( p_h(t, x, y) \) is the elementary solution of (1.1) with \( \mathcal{G}_{s,m,k} \) replaced by \( \mathcal{G}_{s_h,m_h,k_h} \), and \( G_h^\alpha(\alpha, x, y) \) is the \( \alpha \)-Green function corresponding to the ODGDO \( \mathcal{G}_{s_h,m_h,k_h} \).

### 3.2 \( h \) transform of \( \mathcal{G}_{s,m,k} \) with \( h \in \mathcal{H}_{s,m,k,\beta} \)

For \( \beta \geq 0 \), let \( h_\beta(\cdot) \) be a positive continuous function on \( I \) satisfying

\[
h_\beta(x) = h_\beta(c_o) + D_\beta h_\beta(c_o) \{s(x) - s(c_o)\} \\
\quad + \int_{\{c_o, x\}} \{s(x) - s(y)\} h_\beta(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.
\]

There exists such a function \( h_\beta(\cdot) \). Indeed, it is represented as a linear combination of \( g_i(\cdot, \beta) \), \( i = 1, 2 \), given in the preceding section. Let \( \mathcal{H}_{s,m,k,\beta} \) be the set of all positive functions \( h_\beta \) satisfying (3.2). It immediately follows from (3.2) that

\[
\mathcal{H}_{s,m,k,\beta} = \mathcal{H}_{s,m,k+\beta m,0}, \quad \beta \geq 0.
\]

For \( h \in \mathcal{H}_{s,m,k,\beta} \) and \( g_i(x, \alpha) \), \( i = 1, 2 \), satisfying (2.5)–(2.9), we set

\[
g_{h,i}(x, \alpha) = g_i(x, \alpha)/h(x), \quad i = 1, 2, \ \alpha \geq 0.
\]

Let \( G_h^\gamma(\gamma, x, y) \) be the \( \gamma \)-Green function corresponding to \( \mathcal{G}_{s_h,m_h,0} \), where \( s_h \) and \( m_h \) are given by (1.3) and (1.4), respectively. Now we obtain the following proposition. Under the assumption \( I_*(m) = I \), the following result is obtained as Proposition 2.2 and Lemma 3.3 of [8]. It is not difficult to see that their proofs are available without the assumption \( I_*(m) = I \). Therefore we omit the proof of the following proposition.

**Proposition 3.2** Let \( \alpha \geq \beta \geq 0 \) and \( h \in \mathcal{H}_{s,m,k,\beta} \).

(i) Let \( i = 1, 2 \). If \( \alpha = \beta \) and \( |s_h(l_i)| = \infty \), then \( g_{h,i}(x, \alpha) \) is a positive constant function on \( I \). If \( \alpha > \beta \) or \( |s_h(l_i)| < \infty \), then \( g_{h,i}(x, \alpha) \) satisfies the following properties.

(i − 1) \( g_{h,1}(x, \alpha) \) is positive and continuous on \( I \).

(i − 2) \( g_{h,1}(x, \alpha) \) is nondecreasing on \( I \) and \( g_{h,2}(x, \alpha) \) is nonincreasing on \( I \).

(i − 3) If \( |s_h(l_i)| < \infty \) or \( |m_h(l_i)| = \infty \), then \( g_{h,i}(l_i, \alpha) = 0 \).

(i − 4) If \( |s_h(l_i)| = \infty \), then \( D_{s_h} g_{h,i}(l_i, \alpha) = 0 \).
\( g_{h,i}(x, \alpha) \) satisfies
\[
g_{h,i}(x, \alpha) = g_{h,i}(c_0, \alpha) + D_{s_h} g_{h,i}(c_0, \alpha) \{ s_h(x) - s_h(c_0) \}
+ (\alpha - \beta) \int_{[c_0,x]} \{ s_h(x) - s_h(y) \} g_{h,i}(y, \alpha) \, dm_h(y),
\]
for \( x \in I \).

(ii) The following (3.5) holds true.
\[
G_s^*(\alpha - \beta, x, y) = G_s^*(\alpha - \beta, y, x) = W(\alpha)^{-1} g_{h,1}(x, \alpha) g_{h,2}(y, \alpha) = G(\alpha, x, y) / h(x) h(y),
\]
for \( l_1 < x \leq y < l_2 \).

Let \( p(t, x, y) \) be the elementary solution of the equation (1.1) and consider \( p_h^*(t, x, y) \) defined by (1.6). By virtue of (2.11) and (3.5), we get the following.

**Proposition 3.3** \( p_h^*(t, x, y) \) is the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( G_{s_h,m,h,0} \), and \( G_h^*(\alpha, x, y) \) is the \( \alpha \)-Green function corresponding to \( G_{s_h,m,h,0} \).

### 3.3 Proof of Proposition 1.1

(i) We set \( \Lambda = \{ h \in M_{s,m}; D_s h(x) \) is a constant function on \( I \} \). It follows from (3.2) that
\[
D_s h(y) - D_s h(x) = \int_{[x,y]} h(z) \{ \beta \, dm(z) + d k(z) \},
\]
for \( h \in H_{s,m,k,\beta} \). Therefore, \( h \) belongs to \( H_{s,m,0,0} \) if and only if \( D_s h \) is a constant function, from which the set \( \{ x \in I; D_s h(x_1) > D_s h(x_2) \) for \( l_1 < x_1 < x_2 < l_2 \} \) is empty and hence it is included in \( I_*(m) \). Thus \( H_{s,m,0,0} \) is included in the set \( \Lambda \). Conversely, \( h \in \Lambda \) if and only if (3.6) holds true for the null killing measure and \( \beta = 0 \), which implies (3.2) with \( h = h_\beta, \) \( k = 0 \) and \( \beta = 0 \). Therefore the set \( \Lambda \) is included in \( H_{s,m,0,0} \).

(ii) Assume that \( k \) is not a null measure or \( \beta > 0 \). Let \( h \in H_{s,m,k,\beta} \). By means of (3.6) we see that \( D_s h(x) < D_s h(y) \) whenever \( x < y \) and \( (x, y] \cap I_*(m) \neq \emptyset \). Therefore \( D_s h \) is increasing on \( I_*(m) \) and it is nondecreasing on \( I \). Thus \( h \) does not belong to \( M_{s,m} \), which implies the second statement. □
3.4 Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Let $u$ be a measurable function on $I$. We show the first statement (i). Let $G^\alpha_0(\alpha, x, y)$ be the $\alpha$-Green function corresponding to $G_{s,m,0}$. Let $h \in \mathcal{M}_{s,m}$ and consider $G^\alpha_h(\alpha, x, y)$ given by (3.1). By means of (2.14) and Proposition 3.1, $u$ belongs to $D(G_{{s,h,m,h},k_h})$ if and only if there is a function $f \in L^2(I, m_h)$ such that

$$u(x) = \int_I G^\alpha_h(\alpha, x, y)f(y) \, dm_h(y), \quad (3.7)$$

or equivalently,

$$u(x) = \frac{1}{h(x)} \int_I G^\alpha(\alpha, x, y)f(y)h(y) \, dm(y). \quad (3.8)$$

We note that $fh$ belongs to $L^2(I, m)$. Therefore (3.8) holds true if and only if $hu$ belongs to $D(G_{s,m,0})$. Proposition 3.1 combined with (3.7) and (3.8) implies

$$u(x) = G^\alpha_{h,\alpha}f(x) = \frac{1}{h(x)} G^\alpha_\alpha(fh)(x),$$

where $G^\alpha_{h,\alpha}$ and $G^\alpha_\alpha$ are the $\alpha$-Green operators corresponding to $G_{s,h,m,k_h}$ and $G_{s,m,0}$, respectively. By means of (2.16),

$$(\alpha - G_{s,h,m,k_h})G^\alpha_{h,\alpha}f = f, \quad (\alpha - G_{s,m,0})G^\alpha_\alpha(fh) = fh,$$

and hence we obtain

$$(\alpha - G_{s,h,m,k_h})u = \frac{1}{h}(\alpha - G_{s,m,0})(hu),$$

that is,

$$G_{s,h,m,k_h}u = \frac{1}{h}G_{s,m,0}(hu).$$

Thus we get the statement (i).

The statement (ii) is obtained in the same way as above. So we omit the proof. \[\square\]

Proof of Corollary 1.3. The first statement (i) immediately follows from Proposition 1.1 and Theorem 1.2.
(ii) Let $h \in \mathcal{M}_{s,m}$. By means of Theorem 1.2, $H^*_h \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$ if and only if $k_h$ is the null measure, and $s_h(x) = As(x) + A_1$, $m_h(x) = A^{-1}m(x) + A_2$ for some positive constant $A$ and $A_i \in \mathbb{R}$, $i = 1, 2$, or equivalently $h$ is a positive constant function.

(iii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. By means of Theorem 1.2, $H^*_h \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s,m,k+\beta m}$ if and only if $k + \beta m$ is the null measure, and $s_h(x) = Bs(x) + B_1$, $m_h(x) = B^{-1}m(x) + B_2$ for some positive constant $B$ and $B_i \in \mathbb{R}$, $i = 1, 2$. This is equivalent to $k$ being the null measure, $\beta = 0$, and $h$ being a positive constant function. \hfill \Box

\section*{3.5 Proof of Theorem 1.4}

(i) Let $h \in \mathcal{M}_{s,m}$. Then $D_{s_h} h^{-1} = -D_s h$ and hence

$$D_{s_h} h^{-1}(y) - D_{s_h} h^{-1}(x) = -D_s h(y) + D_s h(x)$$

$$= -\int_{(x,y]} dD_s h(z) = \int_{(x,y]} h^{-1}(z) dk_h(z).$$

This shows that $h^{-1}$ belongs to $\mathcal{H}_{s_h,m_h,k_h,0}$. By means of Theorem 1.2 (ii), $H^*_{h^{-1}} \mathcal{G}_{s,m,k,h} = \mathcal{G}_{s,m,0}$, from which $H^*_{h^{-1}} H^*_h \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0}$.

(ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then $D_{s_h} h^{-1} = -D_s h$, and by means of (3.6),

$$D_{s_h} h^{-1}(y) - D_{s_h} h^{-1}(x) = -D_s h(y) + D_s h(x)$$

$$= -\int_{(x,y]} h(z) \{\beta dm(z) + dk(z)\}. \tag{3.9}$$

Therefore $D_{s_h} h^{-1}$ is nonincreasing on $I$ and the set $\Lambda = \{x \in I : D_{s_h} h^{-1}(x_1) > D_{s_h} h^{-1}(x_2) \text{ for } l_1 < x_1 < x < x_2 < l_2 \}$ is included in $I_s(m) \cup I_s(k)$. Since $I_s(m) \cup I_s(k) = I_s(m_h) = I_x(m_h)$, we get $h^{-1} \in \mathcal{M}_{s_h,m_h}$. Calculating the right hand side of (1.5) with $s$ and $h$ replaced by $s_h$ and $h^{-1}$, respectively, we see by virtue of (3.9) that

$$-\int_{(c,x]} h^{-1}(y) dD_{s_h} h^{-1}(y) = \int_{(c,x]} \{\beta dm(y) + dk(y)\}.$$ 

Combining this with Theorem 1.2 (i), we get $H^*_{h^{-1}} \mathcal{G}_{s_h,m_k,0} = \mathcal{G}_{s,m,k+\beta m}$, from which $H^*_{h^{-1}} H^*_h \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s,m,k+\beta m}$. \hfill \Box

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4 Global properties of ODGDOs and their $h$ transform

In this section we show Theorem 1.5. First we prepare two lemmas on properties of $h \in \mathcal{M}_{s,m}$ and $h \in \mathcal{H}_{s,m,k,\beta}$.

**Lemma 4.1** Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$. If $h \in \mathcal{M}_{s,m}$ and $k_h$ is the null measure, then $h$ is a positive constant function. If $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$, then $h$ is a positive constant function and $\beta = 0$.

**Proof.** Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$.

(i) Suppose that $h \in \mathcal{M}_{s,m}$ and $k_h$ is the null measure. By means of (1.5), $D_s h$ is a constant function. Therefore $h(x)$ is represented as $h(x) = C_1 s(x) + C_2$ for some $C_i \in \mathbb{R}$, $i = 1, 2$. Since $s(l_1) = -\infty$, $s(l_2) = \infty$ and $h(x)$ is a positive function, we get $C_1 = 0$ and $C_2 > 0$.

(ii) Suppose that $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$. By using Lemma 3.2 (i) of [8], we see that $h(l_i) < \infty$, $i = 1, 2$. Note that $h(x)$ is represented as $h(x) = C_3 g_1(x, \beta) + C_4 g_2(x, \beta)$, where $g_i(x, \beta)$, $i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$ and $k = 0$. If $\beta > 0$, by means of Proposition 2.1 (i), $g_i(l_j, \beta) < \infty$ for $i = 1, 2$, and $g_i(l_j, \beta) = \infty$ for $i \neq j$. Combining these with $h(l_i) < \infty$ for $i = 1, 2$, we see $C_3 = C_4 = 0$, and hence $h(x) = 0$. This contradicts $h(x) > 0$ on $I$. Thus $\beta = 0$. Noting $s(l_1) = -\infty$, $s(l_2) = \infty$ and Proposition 2.1 (ii), we obtain that $h$ is a positive constant function. \hfill $\square$

**Lemma 4.2** (i) Let $h \in \mathcal{M}_{s,m}$. If $s(l_1) > -\infty$ or $s(l_2) < \infty$, then $s_h(l_1) > -\infty$, or $s_h(l_2) < \infty$, or $k_h$ is not a null measure.

(ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. If $s(l_1) > -\infty$, or $s(l_2) < \infty$, or $k$ is not a null measure, then $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$.

**Proof.** (i) Let $h \in \mathcal{M}_{s,m}$, and $s(l_1) > -\infty$ or $s(l_2) < \infty$. It is enough to show that $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$ in the case that $k_h$ is the null measure. Since $k_h$ is the null measure, $D_s h$ is a constant function by virtue of (1.5).

Suppose $s(l_1) > -\infty$. By means of Lemma 2.1 (ii) of [4], we have $h(l_1) \in [0, \infty)$. If $h(l_1) \in (0, \infty)$, then $s_h(l_1) > -\infty$. Let $h(l_1) = 0$. Noting that $D_s h$ is a constant function, we see that $h(x) = C_1 \{s(x) - s(l_1)\}$ for some $C_1 > 0$. (1.3) coupled with this implies $s_h(l_2) < \infty$.\[15]
In the same way as above, we get \( s_h(l_1) > -\infty \) or \( s_h(l_2) < \infty \) when \( s(l_2) < \infty \).

(ii) Let \( h \in \mathcal{H}_{s,m,k,\beta} \) and assume that \( s(l_1) > -\infty \), or \( s(l_2) < \infty \), or \( k \) is not a null measure. Note \( h(x) \) is represented as \( h(x) = C_2g_1(x,\beta) + C_3g_2(x,\beta) \), where \( g_i(x,\beta), \ i = 1,2, \) are functions satisfying (2.5)–(2.9) with \( \alpha = \beta \). We divide the proof into four cases.

Case 1: \( h(l_1) = \infty \). Then \( s_h(l_1) > -\infty \) by virtue of Lemma 3.2 (i) of [8].

Case 2: \( h(l_1) \in (0,\infty) \) and \( s(l_1) > -\infty \). Then \( s_h(l_1) > -\infty \) by means of (1.3).

Case 3: \( h(l_1) \in (0,\infty) \) with \( s(l_1) = -\infty \). Since \( g_2(l_1,\beta) = \infty \) for \( \beta \geq 0 \) by virtue of Proposition 2.1, we get \( C_3 = 0 \) and \( h(x) = C_2g_1(x,\beta) \).

(3-1) Let \( s(l_2) < \infty \). Then

\[
    s_h(l_2) = \int_{(c_o,l_2)} h(y)^{-2} ds(y) \leq C_2^{-2} g_1(c_o,\beta)^{-2} \{s(l_2) - s(c_o)\} < \infty.
\]

(3-2) Let \( s(l_2) = \infty \). Since \( k \) is not a null measure, we have, by virtue of Proposition 2.1 (i), \( h(l_2) = C_2g_1(l_2,\beta) = \infty \) for \( \beta \geq 0 \). Therefore \( s_h(l_2) < \infty \) by virtue of Lemma 3.2 (i) of [8].

Case 4: \( h(l_1) = 0 \). Then \( h(x) \) is represented as \( h(x) = C_2g_1(x,\beta) \), because of \( g_2(l_1,\beta) > 0 \). Thus we have \( s_h(l_2) < \infty \) in the same way as in (3-1) and (3-2).

\( \square \)

**Proof of Theorem 1.5.** (i) Let \( \mathcal{G}_{s,m,0} \in \Phi^R \). Hence \( s(l_1) = -\infty \) and \( s(l_2) = \infty \).

Suppose that \( H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s_h,m_h,k_h} \in \Phi^R \) for some \( h \in \mathcal{M}_{s,m} \). Then \( k_h \) is the null measure. By virtue of Lemma 4.1, \( h \) is a positive constant function.

Suppose that \( H_h^o \mathcal{G}_{s,m,\beta m} = \mathcal{G}_{s_h,m,0} \in \Phi^R \) for some \( h \in \mathcal{H}_{s,m,0,\beta} \). Then \( s_h(l_1) = -\infty \) and \( s_h(l_2) = \infty \). By virtue of Lemma 4.1, \( h \) is a positive constant function and \( \beta = 0 \).

(ii) The former statement follows from Corollary 1.3. The later follows from Lemma 4.1.

(iii) (1) Let \( \mathcal{G}_{s,m,0} \in \Phi^T \) and \( h \in \mathcal{M}_{s,m} \). Since \( s(l_1) > -\infty \) or \( s(l_2) < \infty \), by means of Lemma 4.2 (i), \( s_h(l_1) > -\infty \) or \( s_h(l_2) < \infty \) or \( k_h \) is not a null measure. Thus \( H_h^o \mathcal{G}_{s,m,0} = \mathcal{G}_{s_h,m,k_h} \in \Phi^T \).

(2) Let \( \mathcal{G}_{s,m,k} \in \Phi^T \) and \( h \in \mathcal{H}_{s,m,k,\beta} \). Noting that \( s(l_1) > -\infty \), or \( s(l_2) < \infty \), or \( k \) is not a null measure, and using Lemma 4.2 (ii), we see
\[ s_h(l_1) > -\infty \text{ or } s_h(l_2) < \infty. \] This implies \( H_h^* \mathcal{G}_{s,m,k+\beta m} = \mathcal{G}_{s_h,m_h,0} \in \Phi^T. \] \[ \square \]

5 Spectral representation of elementary solution

Let \( \mathcal{G}_{s,m,k} \) be an ODGDO and \( p(t,x,y) \) be the elementary solution of the equation (1.1). We assume that \( I_*(m) = I \) and \( l_1 \) is not \((s,m,k)\)-natural. Further we assume that \( p(t,x,y) \) is represented as (1.7). Namely,

\[ p(t,x,y) = \int_{[0,\infty)} e^{-\lambda} \psi(x,\lambda) \psi(y,\lambda) d\sigma(\lambda), \quad t > 0, \ x, y \in I, \]

where \( d\sigma(\lambda) \) is a Borel measure on \([0,\infty)\); \( \sigma(\{0\}) = 0 \) if \( l_1 \) is \((s,m,k)\)-regular or exit. Further \( \psi(x,\lambda), \ x \in I, \ \lambda \geq 0, \) is the solution of the following integral equation.

\[ \psi(x,\lambda) = s(x) - s(l_1) + \int_{(l_1,x]} \{s(x) - s(y)\} \psi(y,\lambda) \{-\lambda \, dm + dk\}, \]

if \( l_1 \) is \((s,m,k)\)-regular or exit. \( \quad (5.1) \)

\[ \psi(x,\lambda) = 1 + \int_{(l_1,x]} \{s(x) - s(y)\} \psi(y,\lambda) \{-\lambda \, dm + dk\}, \]

if \( l_1 \) is \((s,m,k)\)-entrance. \( \quad (5.2) \)

In the following we fix an \( h \in H_{s,m,k,\beta} \) arbitrarily. Then \( \mathcal{G}_{s_h,m_h,0} \) is an ODGDO and \( p^*_h(t,x,y) \) given by (1.6) is the elementary solution of the equation (1.1) with \( \mathcal{G}_{s_h,m_h,0} \) in place of \( \mathcal{G}_{s,m,k} \). We set \( \psi_h(x,\lambda) = \psi(x,\lambda)/h(x) \). By virtue of (1.6) and (1.7), we obtain

\[ p^*_h(t,x,y) = e^{-\beta t} p(t,x,y) \]

\[ = \int_{[\beta,\infty)} e^{-\lambda} \psi_h(x,\lambda - \beta) \psi_h(y,\lambda - \beta) \, d\sigma(\lambda - \beta). \quad (5.3) \]

In the same way as in the proof of Lemma 3.3 of [8], we see that \( \psi_h(x,\lambda) \) satisfies the following.

\[ D_{s_h} \psi_h(y,\lambda) - D_{s_h} \psi_h(x,\lambda) = -(\lambda + \beta) \int_{(x,y]} \psi_h(z,\lambda) \, dm_h(z), \quad (5.4) \]
for $l_1 < x < y < l_2$. We note the following.

**Lemma 5.1** (i) Let $l_1$ be $(s, m, k)$-regular or exit. Then $h(l_1) \in [0, \infty)$ and

\[
\psi_h(l_1, \lambda) = D_s h(l_1)^{-1} \in (0, \infty), \quad D_{s_{h}} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) = 0; \tag{5.5}
\]

\[
\psi_h(l_1, \lambda) = 0, \quad D_{s_{h}} \psi_h(l_1, \lambda) = h(l_1) \in (0, \infty), \quad \text{if } h(l_1) \in (0, \infty). \tag{5.6}
\]

(ii) Let $l_1$ be $(s, m, k)$-entrance. Then $h(l_1) \in (0, \infty)$ and

\[
\psi_h(l_1, \lambda) = h(l_1)^{-1} \in (0, \infty), \quad D_{s_{h}} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) \in (0, \infty); \tag{5.7}
\]

\[
\psi_h(l_1, \lambda) = 0, \quad D_{s_{h}} \psi_h(l_1, \lambda) = -D_s h(l_1) \in (0, \infty), \quad \text{if } h(l_1) = \infty. \tag{5.8}
\]

**Proof.** Note that $h$ is represented as $h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)$, where $g_i(x, \beta), i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$.

(i) Let $l_1$ be $(s, m, k)$-regular or exit. By means of Theorem 1.1 of [7], $0 \leq h(l_1) < \infty$.

First we consider the case $h(l_1) = 0$, and hence $C_1 > 0$ and $C_2 = 0$, by virtue of Proposition 2.1. Since $D_s h(l_1) = C_1 D_s g_1(l_1, \beta) \in (0, \infty)$, we get

\[
\psi_h(l_1, \lambda) = \lim_{x \to l_1} \frac{\psi(x, \lambda)}{h(x)} = \lim_{x \to l_1} \frac{D_s \psi(x, \lambda)}{D_s h(x)} = \frac{1}{D_s h(l_1)} \in (0, \infty),
\]

\[
D_{s_{h}} \psi_h(l_1, \lambda) = \lim_{x \to l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = 0.
\]

Thus we obtain (5.5).

Next we consider the case $0 < h(l_1) < \infty$. Then $C_2 > 0$ and $\psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0$. If $l_1$ is $(s, m, k)$-regular, then by means of Proposition 2.1,

\[
D_s h(l_1) = C_1 D_s g_1(l_1, \beta) + C_2 D_s g_2(l_1, \beta) \in \mathbb{R}
\]

and hence

\[
D_{s_{h}} \psi_h(l_1, \lambda) = \lim_{x \to l_1} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = h(l_1) \in (0, \infty).
\]

If $l_1$ is $(s, m, k)$-exit, then by means of Proposition 2.1

\[
\lim_{x \to l_1} \frac{\psi(x, \lambda)}{g_1(x, \beta)} = \lim_{x \to l_1} \frac{D_s \psi(x, \lambda)}{D_s g_1(x, \beta)} = \frac{1}{D_s g_1(l_1, \beta)} \in (0, \infty),
\]

\[
\lim_{x \to l_1} \psi(x, \lambda) D_s h(x) = \frac{1}{D_s g_1(l_1, \beta)} \lim_{x \to l_1} g_1(x, \beta) \{C_1 D_s g_1(x, \beta) + C_2 D_s g_2(x, \beta)\} = 0.
\]
Therefore we arrive at

\[
D_s \psi_h(l_1, \lambda) = \lim_{x \to l_1} \{h(x)D_s \psi(x, \lambda) - \psi(x, \lambda)D_s h(x)\} = h(l_1) \in (0, \infty).
\]

Thus we have (5.6).

(ii) Let \(l_1\) be \((s, m, k)\)-entrance. By means of Theorem 1.1 of [7], \(0 < h(l_1) \leq \infty\). First we consider the case \(0 < h(l_1) < \infty\), and hence \(C_1 > 0\) and \(C_2 = 0\) by virtue of Proposition 2.1. Then

\[
\psi_h(l_1, \lambda) = \frac{1}{h(l_1)} = \frac{1}{C_1 g_1(l_1, \beta)} \in (0, \infty),
\]

\[
D_s \psi_h(l_1, \lambda) = h(l_1) D_s \psi(l_1, \lambda) - \psi(l_1, \lambda) D_s h(l_1) = -C_1 D_s g_1(l_1, \beta) = 0.
\]

These show (5.7).

Next we consider the case \(h(l_1) = \infty\). Then \(C_2 > 0\) and \(\psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0\). We show that

\[
D_s \psi_h(l_1, \lambda) = -D_s h(l_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty). \tag{5.9}
\]

Here is the proof of (5.9). Since \(l_1\) is \((s, m, k)\)-entrance, for any positive \(\varepsilon\) there exists an \(r \in I\) such that

\[
\{s(r) - s(x)\} \int_{(l_1, x]} (\lambda dm(z) + dk(z)) \\
\leq \int_{(l_1, x]} \{s(r) - s(z)\} (\lambda dm(z) + dk(z)) \\
\leq \int_{(l_1, r]} \{s(r) - s(z)\} (\lambda dm(z) + dk(z)) < \varepsilon, \tag{5.10}
\]

for \(l_1 < x < r\). It is easy to derive the following estimate from (5.2).

\[
|D_s \psi(x, \lambda)| \leq \int_{(l_1, x]} (\lambda dm(z) + dk(z)) \exp \left\{ \int_{(l_1, x]} (\lambda dm(z) + dk(z)) \int_{(z, x]} ds(y) \right\}.
\]

Combining this with (5.10), we find

\[
\lim_{x \to l_1} \sup_{s(r) - s(x)} |D_s \psi(x, \lambda)| < \varepsilon.
\]
Therefore
\[ \limsup_{x \to l_1} h(x) |D_s \psi(x, \lambda)| \leq \varepsilon \lim_{x \to l_1} \frac{h(x)}{s(r) - s(x)} = -\varepsilon D_s h(l_1) = -\varepsilon C_2 D_s g_2(l_1, \beta). \]

Since \( D_s g_2(l_1, \beta) \in (-\infty, 0) \) by virtue of Proposition 2.1, letting \( \varepsilon \downarrow 0 \) leads us to \( \lim_{x \to l_1} h(x) D_s \psi(x, \lambda) = 0 \), and

\[ D_{sh} \psi_h(l_1, \lambda) = \lim_{x \to l_1} \{ h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x) \} = -D_s h(l_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty), \]

which shows (5.9). Thus we obtain (5.8).

Now we define \( \psi^*_h(x, \lambda), \ x \in I, \ \lambda \geq \beta, \) and \( d\sigma^*_h(\lambda), \ \lambda \geq \beta, \) as follows.

Case 1. \( l_1 \) is \( (s, m, k) \)-regular or exit. If \( h(l_1) = 0 \), then
\[ \psi^*_h(x, \lambda) = D_s h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \]
\[ d\sigma^*_h(\lambda) = \{ D_s h(l_1) \}^{-2} d\lambda \sigma(\lambda - \beta). \]

If \( h(l_1) \in (0, \infty) \), then
\[ \psi^*_h(x, \lambda) = \{ h(l_1) h(x) \}^{-1} \psi(x, \lambda - \beta), \]
\[ d\sigma^*_h(\lambda) = h(l_1)^2 d\lambda \sigma(\lambda - \beta). \]

Case 2. \( l_1 \) is \( (s, m, k) \)-entrance. If \( h(l_1) \in (0, \infty) \), then
\[ \psi^*_h(x, \lambda) = h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \]
\[ d\sigma^*_h(\lambda) = h(l_1)^{-2} d\lambda \sigma(\lambda - \beta). \]

If \( h(l_1) = \infty \), then
\[ \psi^*_h(x, \lambda) = \{-D_s h(l_1) h(x) \}^{-1} \psi(x, \lambda - \beta), \]
\[ d\sigma^*_h(\lambda) = \{ D_s h(l_1) \}^2 d\lambda \sigma(\lambda - \beta). \]

We note that \( \sigma^*_h(\{ \beta \}) = 0 \) in Case 1, but \( \sigma^*_h(\{ \beta \}) \geq 0 \) in Case 2.

By means of Theorem 1.1 of [7], \( l_1 \) is not \( (s_h, m_h, 0) \)-natural. More precisely, in the case \( l_1 \) is \( (s, m, k) \)-regular [resp. exit],
- \( l_1 \) is \( (s, m, k) \)-regular [resp. exit] if \( h(l_1) \in (0, \infty) \),
- \( l_1 \) is \( (s_h, m_h, 0) \)-entrance if \( h(l_1) = 0 \).
in the case \( l_1 \) is \((s_h, m_h, 0)\)-entrance,

\[
\begin{align*}
\text{\( l_1 \) is \((s_h, m_h, 0)\)-entrance} & \quad \text{if} \ h(l_1) \in (0, \infty), \\
\text{\( l_1 \) is \((s_h, m_h, 0)\)-regular} & \quad \text{if} \ h(l_1) = \infty \text{ and } |m_h(l_1)| < \infty, \\
\text{\( l_1 \) is \((s_h, m_h, 0)\)-exit} & \quad \text{if} \ h(l_1) = \infty \text{ and } |m_h(l_1)| = \infty.
\end{align*}
\]

Therefore (5.3), (5.4) and Lemma 5.1 lead us to the following result.

**Proposition 5.2** \( \psi_h^*(x, \lambda) \) satisfies (5.1) or (5.2) with \( s, m \) and \( k \) replaced by \( s_h, m_h \) and \( 0 \), respectively. 

\[d\sigma_h^*(\lambda)\] is a Borel measure on \([\beta, \infty)\). 

\[p_h^*(t, x, y)\] is represented as

\[
p_h^*(t, x, y) = \int_{(0, \infty)} e^{-\lambda \psi_h^*(x, \lambda) \psi_h^*(y, \lambda)} d\sigma_h^*(\lambda), \quad t > 0, \ x, y \in I. \quad (5.19)
\]

### 6 Examples

**Example 6.1** First we consider the following ODGDO \( G_{s,m,k} \) on \( I = (0, \infty) \) with constant coefficients.

\[
G_{s,m,k} = a \frac{d^2}{dx^2} + b \frac{d}{dx} - c, \quad (6.1)
\]

where \( a > 0, \ b \in \mathbb{R} \) and \( c \geq 0 \). We may set

\[
ds(x) = e^{-(b/a)x} dx, \ dm(x) = a^{-1}e^{(b/a)x} dx, \ dk(x) = (c/a)e^{(b/a)x} dx. \quad (6.2)
\]

The end point \( 0 \) is \((s, m, k)\)-regular and the end point \( \infty \) is \((s, m, k)\)-natural. For \( \alpha \geq 0 \), we set

\[
\lambda_i(\alpha) = \frac{1}{2a} \left\{ (-1)^i b + \sqrt{b^2 + 4a(c + \alpha)} \right\}, \quad i = 1, 2.
\]

The \( \alpha \)-Green function \( G(\alpha, x, y) \) corresponding to \( G_{s,m,k} \) is given by the following.

\[
G(\alpha, x, y) = G(\alpha, y, x) = \begin{cases} 
(\lambda_1(\alpha) + \lambda_2(\alpha))^{-1} g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha > 0, \\
g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha = 0,
\end{cases}
\]

\[21\]
for \(0 \leq x \leq y < \infty\), where
\[
g_1(x, \alpha) = \begin{cases} 
  e^{\lambda_1(x)} - e^{\lambda_2(x)}, & \text{if } b^2 + c + \alpha > 0, \\
  x, & \text{if } b^2 + c + \alpha = 0,
\end{cases}
\]
\[
g_2(x, \alpha) = \begin{cases} 
  e^{-\lambda_2(x)}, & \text{if } b^2 + c + \alpha > 0, \\
  1, & \text{if } b^2 + c + \alpha = 0.
\end{cases}
\]

The elementary solution \(p(t, x, y)\) is given by
\[
p(t, x, y) = \frac{1}{2} \sqrt{\frac{a}{\pi t}} \exp \left\{ -\frac{b}{2a}(x + y) - At \right\} \left\{ e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right\} = \int_{\Lambda} e^{-\lambda \psi(x, \lambda) \psi(y, \lambda)} d\sigma(\lambda),
\]
where \(A = b^2/4a + c\) and
\[
\psi(x, \lambda) = e^{-\lambda \psi(x, \lambda) \psi(y, \lambda)} d\sigma(\lambda),
\]
for \(x \in I\) and \(\lambda > A\).

Let \(h \in H_{s, m, k, \beta}\). Then the \(h\) transform of \(G_{s, m, k}\) is reduced to the following.
\[
G_{s, m, k, 0} = a \frac{d^2}{dx^2} + \left( b + 2a \frac{h'(x)}{h(x)} \right) \frac{d}{dx}.
\]

(1) Assume \(h(0) \in (0, \infty)\). Then \(h\) is represented as \(h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)\) for some constants \(C_1 \geq 0\) and \(C_2 > 0\). The end point 0 is \((s_h, m_h, 0)\)-regular and we get (5.19), where by means of (5.13) and (5.14),
\[
\psi^*_h(x, \lambda) = \{ C_2 h(x) \}^{-1} \psi(x, \lambda - \beta), \quad d\sigma^*_h(\lambda) = C_2^2 d\sigma(\lambda - \beta).
\]

(2) Assume \(h(0) = 0\). Then \(h(x) = C_1 g_1(x, \beta)\) with \(C_1 > 0\). The end point 0 is \((s_h, m_h, 0)\)-entrance and we get (5.19), where by means of (5.11) and (5.12),
\[
\psi^*_h(x, \lambda) = \tilde{C} h(x)^{-1} \psi(x, \lambda - \beta), \quad d\sigma^*_h(\lambda) = \tilde{C}^{-2} d\sigma(\lambda - \beta).
\]
Here
\[ \tilde{C} = D_s h(0) = \begin{cases} 
C_1(\lambda_1(\beta) + \lambda_2(\beta)), & \text{if } b^2 + c + \beta > 0, \\
C_1, & \text{if } b^2 + c + \beta = 0.
\end{cases} \]

Especially, if \( b = c = 0 \) and \( C_1 = 1 \), then \( \lambda_i(\beta) = \sqrt{\beta/4} \), \( i = 1, 2 \), and
\[ h(x) = \begin{cases} 
2 \sinh(\sqrt{\beta/4} x), & \text{if } \beta > 0, \\
x, & \text{if } \beta = 0.
\end{cases} \]

Therefore (6.3) and (5.19) are reduced to the following. If \( \beta > 0 \), then
\[ G_{s,h,m} = a \frac{d^2}{dx^2} + 2a \beta \frac{\cosh(\sqrt{\beta/4} x)}{\sinh(\sqrt{\beta/4} x)} \frac{d}{dx}, \]
\[ p_h(t, x, y) = \frac{1}{8} \left[ \left( e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right) / (\sinh(\sqrt{\beta/4} x) \sinh(\sqrt{\beta/4} y)) \right] e^{-\lambda^* \psi_h(x, \lambda) \psi_h(y, \lambda)} d\sigma^*_h(\lambda), \]
where
\[ \psi_h(x, \lambda) = \sqrt{\frac{\beta}{\lambda - \beta}} \sin(\sqrt{\lambda/4} x) / \sinh(\sqrt{\beta/4} x), \]
\[ d\sigma^*_h(\lambda) = \frac{1}{4\pi} \sqrt{\lambda/\beta} \sqrt{\beta/\alpha} d\lambda. \]

If \( \beta = 0 \), then
\[ G_{s,h,m} = a \frac{d^2}{dx^2} + \frac{2a}{x} \frac{d}{dx}, \quad (6.5) \]
\[ p_h(t, x, y) = \frac{1}{2} \left[ \frac{1}{\pi t xy} \left( e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right) \right] e^{-\lambda^* \psi_h(x, \lambda) \psi_h(y, \lambda)} d\sigma^*_h(\lambda), \]
where
\[ \psi_h(x, \lambda) = \frac{\sin(\sqrt{\lambda/4} x)}{\sqrt{\lambda/4} x}, \quad d\sigma^*_h(\lambda) = \frac{1}{\pi} \sqrt{\lambda/\alpha} d\lambda. \]

When \( \alpha = 1/2 \), (6.5) is the generator of 3-dimensional Bessel process which we denote by \( \tilde{G} \), that is,
\[ \tilde{G} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx}. \]
Since $\tilde{h}(x) = 1/x$ satisfies $\tilde{G}\tilde{h} = 0$, we can consider the $h$ transform of $\tilde{G}$ with respect to $\tilde{h}(x) = 1/x$, which we denote by $G$. Then we get

$$G = \frac{1}{2} \frac{d^2}{dx^2}.$$ 

This is the generator of Brownian motion on $(0, \infty)$. The above observation shows that $G$ is $h$ transformed to $\tilde{G}$ in terms of $h(x) = x$, and $\tilde{G}$ is $h$ transformed to $G$ in terms of $\tilde{h}(x) = 1/x$. This fact is also obtained by means of Theorem 1.4.

(3) We denote $G_{s,m,k}$ with $b^2 + c > 0$ [resp. $b^2 + c = 0$] by $G^{(1)}$ [resp. $G^{(2)}$]. Fix an $\beta \geq 0$ arbitrarily, and set $h^{(1)}(x) = e^{\lambda_1(\beta)x}$. Since $G^{(1)}h^{(1)} = \beta h^{(1)}$, we can consider an $h$ transform of $G^{(1)}$ with respect to $h^{(1)}$ which is denoted by $\tilde{G}^{(1)}$, that is,

$$\tilde{G}^{(1)} = a \frac{d^2}{dx^2} + \left(b + 2a \frac{h^{(1)'}}{h^{(1)}}\right) \frac{d}{dx} \frac{1}{x} = a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}. \quad (6.7)$$

We put $\kappa = b^2/4a + c + \beta$ and $h^{(2)}(x) = e^{\sqrt{\kappa/a}x}$. Since $G^{(2)}h^{(2)} = \kappa h^{(2)}$, we can consider an $h$ transform of $G^{(2)}$ with respect to $h^{(2)}$ which is denoted by $\tilde{G}^{(2)}$, that is,

$$\tilde{G}^{(2)} = a \frac{d^2}{dx^2} + a \frac{h^{(2)'}}{h^{(2)}} \frac{d}{dx} \frac{1}{x} = a \frac{d^2}{dx^2} + 2a \sqrt{\kappa/a} \frac{d}{dx} \frac{1}{x} = a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}. \quad (6.8)$$

We find that $G^{(1)} \neq G^{(2)}$, but $\tilde{G}^{(1)} = \tilde{G}^{(2)}$. This shows that $h$ transform of ODGDO is not one-to-one correspondence.

**Example 6.2** Let us consider the following ODGDO $G_{(\kappa)}$ on $I = (0, \infty)$.

$$G_{(\kappa)} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\kappa}{2x} \frac{d}{dx},$$
where $\kappa$ is a real number. $\mathcal{G}(0)$ is the generator of Brownian motion on $(0, \infty)$, and $\mathcal{G}(\kappa)$ is the generator of $\kappa + 1$ dimensional Bessel process for $\kappa \in \mathbb{N}$. We may set the scale function $s(\kappa)$ and the speed measure $m(\kappa)$ as follows.

$$ds(\kappa)(x) = x^{-\kappa} \, dx, \quad dm(\kappa)(x) = 2x^\kappa \, dx.$$ 

The end point 0 is $(s(\kappa), m(\kappa), 0)$-exit, or regular, or entrance according to $\kappa \leq -1$, or $-1 < \kappa < 1$, or $1 \leq \kappa$. The end point $\infty$ is always natural. The $\alpha$-Green function $G(\alpha, x, y)$ corresponding to $G(\kappa)$ is given by the following.

$$G(\kappa)(\alpha, x, y) = G(\kappa)(\alpha, y, x) = \begin{cases} (xy)^\mu I_{|\mu|} \left( \sqrt{2\alpha} x \right) K_{|\mu|} \left( \sqrt{2\alpha} y \right), & \text{if } \alpha > 0, \\ (2\mu)^{-1} x^{2\mu}, & \text{if } \kappa < 1, \alpha = 0, \end{cases}$$

for $0 < x \leq y < \infty$, where $I_\nu$ and $K_\nu$ are the modified Bessel functions defined by the following.

$$I_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\Gamma(\nu + n + 1)}, \quad \nu > -1,$$

$$K_\nu(x)(x) = K_{-\nu}(x) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu \pi}, & \nu \in \mathbb{Z}, \\ \frac{(-1)^\nu}{2} \lim_{\xi \to \nu} \left( \frac{\partial I_{-\xi}(x)}{\partial \xi} - \frac{\partial I_\xi(x)}{\partial \xi} \right), & \nu \notin \mathbb{Z}. \end{cases}$$

The elementary solution $p(\kappa)(t, x, y)$ is given as follows.

$$p(\kappa)(t, x, y) = \frac{1}{2t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} (xy)^\mu I_{|\mu|} \left( \frac{xy}{t} \right), \quad t, x, y > 0. \quad \text{(6.9)}$$

We note the following formula (see [6], p.200).

$$\int_0^\infty e^{-\lambda t} J_\nu(\sqrt{\lambda} x) J_\nu(\sqrt{\lambda} y) \, d\lambda = \frac{1}{t} \exp \left\{ -\frac{x^2 + y^2}{4t} \right\} J_\nu \left( \frac{xy}{2t} \right), \quad \text{(6.10)}$$

for $t, x, y > 0$ and $\nu > -1$, where $J_\nu$ is the Bessel function defined by

$$J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(\nu + n + 1)}.$$
By means of (6.9) and (6.10), we get the following representation.

\[ p_{(\kappa)}(t, x, y) = \int_{0}^{\infty} e^{-\lambda t} \psi_{(\kappa)}(x, \lambda) \psi_{(\kappa)}(y, \lambda) \sigma_{(\kappa)}(\lambda) \, d\lambda, \]

where

\[ \psi_{(\kappa)}(x, \lambda) = C_{(\kappa)}(\lambda) x^{\mu} J_{|\mu|} (\sqrt{2\lambda} x), \quad \sigma_{(\kappa)}(\lambda) = C_{(\kappa)}(\lambda)^{-2}, \]

and \( C_{(\kappa)}(\lambda) \) is a positive number satisfying

\[ \lim_{x \to 0} \psi_{(\kappa)}(x, \lambda) = 1 \text{ if } \kappa \geq 1, \]

\[ \lim_{x \to 0} D_{s_{(\kappa)}} \psi_{(\kappa)}(x, \lambda) = 1 \text{ if } \kappa < 1, \]

so that,

\[ C_{(\kappa)}(\lambda) = \begin{cases} 2^{-1} (\lambda/2)^{-\mu/2} \Gamma (\mu), & \text{if } \kappa < 1, \\ (\lambda/2)^{-|\mu|/2} \Gamma (1 + |\mu|), & \text{if } \kappa \geq 1. \end{cases} \]

For \( \beta > 0 \), we set

\[ h_{(\kappa)}(x) = \left( \sqrt{2\beta} x \right)^{\mu} K_{|\mu|} \left( \sqrt{2\beta} x \right). \quad (6.11) \]

The function \( h_{(\kappa)} \) is positive and satisfies \( G_{(\kappa)} h_{(\kappa)} = \beta h_{(\kappa)} \). We denote by \( \widetilde{G}_{(\kappa)} \) the transform of \( G_{(\kappa)} \) with respect to \( h_{(\kappa)} \), that is,

\[ \widetilde{G}_{(\kappa)} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{\kappa}{2x} + \frac{h'_{(\kappa)}(x)}{h_{(\kappa)}(x)} \right) \frac{d}{dx} \quad + \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1}{2x} + \sqrt{2\beta} \frac{K'_{|\mu|}(\sqrt{2\beta} x)}{K_{|\mu|}(\sqrt{2\beta} x)} \right) \frac{d}{dx}. \quad (6.12) \]

Since \( h_{(\kappa)}(0) \in (0, \infty) \) [resp. = \( \infty \)] if \( \kappa \in (-\infty, 1) \) [resp. \( \in [1, \infty) \)], and \( |\tilde{m}_{(\kappa)}(0)| \in [0, \infty) \) [resp. = \( \infty \)] if \( \kappa \in [1, 3) \) [resp. \( \in [3, \infty) \)], the end point 0 is \( (\tilde{s}_{(\kappa)}, \tilde{m}_{(\kappa)}, 0) \)-regular or exit according to \( \kappa < 3 \) or \( \kappa \geq 3 \), where

\[ ds_{(\kappa)}(x) = h_{(\kappa)}(x)^{-2} x^{-\kappa} \, dx, \quad d\tilde{m}_{(\kappa)}(x) = 2 h_{(\kappa)}(x)^2 x^\kappa \, dx. \quad (6.13) \]
We denote by \( \tilde{p}(\kappa)(t, x, y) \) the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( \tilde{G}(\kappa) \). Then we get the following.

\[
\tilde{p}(\kappa)(t, x, y) = e^{-\beta t} p(\kappa)(t, x, y) / \tilde{F}(\kappa),
\]

\[
= \frac{1}{2(2\beta)^{\frac{3}{2}}} \exp \left\{ -\beta t - \frac{x^2 + y^2}{2t} \right\} \left[ K_{|\kappa|}(\sqrt{2\beta} x)K_{|\kappa|}(\sqrt{2\beta} y) \right] \left[ \frac{\tilde{\psi}(\kappa)(x, \lambda)}{\tilde{\sigma}(\kappa)(\lambda)} \right] d\lambda,
\]

(6.14)

where \( \tilde{\psi}(\kappa)(x, \lambda) \) and \( \tilde{\sigma}(\kappa)(\lambda) \) are obtained by means of (5.13), (5.14), (5.17) and (5.18).

We now take \( \kappa_1 \in (-\infty, 1) \) and put \( \kappa_2 = 2 - \kappa_1 \in (1, \infty) \). Then \( |(1 - \kappa_1)/2| = |(1 - \kappa_2)/2| \). We thus find that \( G(\kappa_1) \neq G(\kappa_2) \), but \( \tilde{G}(\kappa_1) = \tilde{G}(\kappa_2) \). This also shows that \( h \) transform of ODGDO is not one-to-one correspondence.

**Example 6.3** Finally we consider the following ODGDO \( G_o \) on \((0, \infty)\).

\[
G_o = \frac{1}{2} d^2 dx^2 - cx^{-2},
\]

(6.15)

where \( c \) is a positive number. We may set

\[
ds_{o}(x) = dx, \quad dm_{o}(x) = 2dx, \quad dk_{o}(x) = 2cx^{-2} dx.
\]

We note that both of the end points 0 and \( \infty \) are \((s_o, m_o, k_o)\)-natural.

Let \( p_o(t, x, y) \) be the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( G_o \). By using Proposition 3.3, we show that \( p_o(t, x, y) \) is given by (6.19) below.

For \( \beta > 0 \), we set

\[
h_o(x) = \sqrt{x} K_{\nu}(\sqrt{2\beta} x),
\]

where \( \nu = \sqrt{2c + 1/4} > 1/2 \). The function \( h_o \) is positive and satisfies \( G_o h_o = \beta h_o \). We denote by \( \tilde{G}_o \) the \( h \) transform of \( G \) with respect to \( h_o \). Then we find

\[
\tilde{G}_o = \frac{1}{2} d^2 dx^2 + \frac{h'_o(x)}{h_o(x)} \frac{d}{dx} = \frac{1}{2} d^2 dx^2 + \left\{ \frac{1}{2x} + \sqrt{2\beta} \frac{K'_{\nu}(\sqrt{2\beta} x)}{K_{\nu}(\sqrt{2\beta} x)} \right\} \frac{d}{dx}.
\]

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This shows that $\widetilde{G}_o$ coincides with $\widetilde{G}_{(\kappa)}$ with $\nu = |\mu| = |(1 - \kappa)/2|$ in the preceding example. It should be noted that we take the scale $\tilde{s}_o$ [resp. $\tilde{s}_{(\kappa)}$] and the speed measure $\tilde{m}_o$ [resp. $\tilde{m}_{(\kappa)}$] corresponding to $\widetilde{G}_o$ [resp. $\widetilde{G}_{(\kappa)}$], where

$$d\tilde{s}_o(x) = h_o(x)^{-2} \, dx, \quad d\tilde{m}_o(x) = 2h_o(x)^2 \, dx.$$  \hfill (6.16)

Let $\tilde{p}_o(t, x, y)$ be the elementary solution of the equation (1.1) with $G_{s,m,k}$ replaced by $\widetilde{G}_o$. Then we have

$$\tilde{p}_o(t, x, y) = e^{-\beta t} p_o(t, x, y)/h_o(x)h_o(y).$$  \hfill (6.17)

Let $\nu = |(1 - \kappa)/2|$. Since $\widetilde{G}_o$ coincides with $\widetilde{G}_{(\kappa)}$, noting (6.16) and (6.13), we see that

$$\tilde{p}_o(t, x, y)h_o(y)^2 = \tilde{p}_{(\kappa)}(t, x, y)h_{(\kappa)}(y)^2y^\kappa. \hfill (6.18)$$

By using (6.9), (6.11), (6.14), (6.17), and (6.18), we obtain the following.

$$p_o(t, x, y) = p_{(\kappa)}(t, x, y) \frac{h_{(\kappa)}(y)}{h_{(\kappa)}(x)} \frac{h_o(x)}{h_o(y)} y^\kappa = \frac{1}{2t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} (xy)^{1/2} I_\nu \left( \frac{xy}{t} \right). \hfill (6.19)$$

**References**


