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<tr>
<td>Authors</td>
<td>Takemura, Tomoko; Tomisaki, Matsuyo</td>
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<tr>
<td>Citation</td>
<td>Takemura Tomoko, Tomisaki Matsuyo; Kyushu Journal of Mathematics, Vol.66(1) pp.171-191</td>
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<td>Issue Date</td>
<td>2012-03</td>
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<td>URL</td>
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$h$ transform of one dimensional generalized diffusion operators

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Abstract

We are concerned with two types of $h$ transform of one dimensional generalized diffusion operators treated by Maeno(2006) and by the second author(2007). We show that these two types of $h$ transform are in inverse relation to each other in some sense. Further we show that a recurrent one dimensional generalized diffusion operator cannot be represented as an $h$ transform of another one dimensional generalized diffusion operator different from the original one. We also consider a spectral representation of elementary solutions corresponding to $h$ transformed one dimensional generalized diffusion operators.

*2010 Mathematics Subject Classification: 60J60, 60G20
Key Words and Phrases: generalized diffusion process, harmonic transformation, recurrence

The second author was partially supported by Grant-in-Aid for Scientific Research (C) 22540132.
1 Introduction

Let $G_{s,m,k}$ be a one dimensional generalized diffusion operator on an interval $I$ with scale function $s$, speed measure $m$, and killing measure $k$ (ODGDO with $(s,m,k)$ for brief). Let $M_{s,m}$ be the set of all positive functions $h$ satisfying $G_{s,m,0}h \leq 0$, where $G_{s,m,0}$ is an ODGDO with $(s,m,0)$ and 0 denotes the null killing measure. For $\beta \geq 0$, let $H_{s,m,k,\beta}$ be the set of all positive functions satisfying $G_{s,m,k}h = \beta h$. In this paper, we are concerned with two types of $h$ transform based on $h \in M_{s,m}$ and $h \in H_{s,m,k,\beta}$.

It is well known that the generator of one dimensional generalized diffusion process (ODGDP for brief) is represented as an ODGDO. In Section 4.3 of [3] K. Itô and H. P. McKean used a method to derive such general form $G_{s,m,k}$ for the generator of ODGDP. Their method is an $h$ transform based on the probability which the sample paths hit one of the end points of the state interval. Following their idea, Maeno [4] considered $h$ transforms of ODGDPs based on $h \in M_{s,m}$ and studied some properties corresponding to $h$ transformed ODGDP. On the other hand, the second author [8] considered $h$ transforms of ODGDOs based on $h \in H_{s,m,k,\beta}$ and obtained some results for $h$ transformed ODGDOs. As we will see in Proposition 1.1 below, the sets $M_{s,m}$ and $H_{s,m,k,\beta}$ are usually disjoint each other. Therefore the results of [4] and [8] are not necessarily derived from each other.

Proposition 1.1 (i) The set $H_{s,m,0,0}$ coincides with the set \{ $h \in M_{s,m}; D_s h(x)$ is a constant function on $I$ \}.

(ii) If $k$ is not a null measure or $\beta > 0$, then $M_{s,m} \cap H_{s,m,k,\beta} = \emptyset$.

Here $D_s h$ is the right derivative of $h$ with respect to the scale function $s$. We show this proposition in Section 3.

Let $p(t,x,y)$ be the elementary solution of the equation

$$\frac{\partial}{\partial t} p(t,x,y) = G_{s,m,k} p(t,x,y), \quad t > 0, \ x, y \in I,$$

in the sense of McKean [5], where $G_{s,m,k}$ is applied to $x$ or $y$.

Let $p^\rho(t,x,y)$ be the elementary solution of the equation (1.1) with $G_{s,m,k}$
replaced by $G_{s,m,0}$. For $h \in \mathcal{M}_{s,m}$, we set
\[ p^h(t, x, y) = p^o(t, x, y)/h(x)h(y), \quad t > 0, \ x, y \in I, \tag{1.2} \]
\[ s_h(x) = \int_{(c_o, x]} h(y)^{-2} ds(y), \tag{1.3} \]
\[ m_h(x) = \int_{(c_o, x]} h(y)^2 dm(y), \tag{1.4} \]
\[ k_h(x) = -\int_{(c_o, x]} h(y) dD_s h(y), \tag{1.5} \]
where $c_o$ is a point of $I$ fixed arbitrarily. Maeno [4] showed that $p^h(t, x, y)$ is the elementary solution of the equation (1.1) with $G_{s,m,k,h}$ in place of $G_{s,m,k}$. Further she studied the asymptotic behavior near the boundaries of $I$ for sample paths of ODGDP with generator $G_{s,m,k,h}$, and gave a precise classification of the states of the boundaries by means of $s$, $m$ and $h$. We call $G_{s,m,k,h}$ the $h$ transform of $G_{s,m,0}$ with $h \in \mathcal{M}_{s,m}$.

We next turn to an $h$ transform treated in [8]. Let $p(t, x, y)$ be the elementary solution of the equation (1.1). For $h \in H_{s,m,k,\beta}$ set
\[ p^h(t, x, y) = e^{-\beta t} p(t, x, y)/h(x)h(y), \quad t > 0, \ x, y \in I. \tag{1.6} \]
Then $p^h(t, x, y)$ is the elementary solution of the equation (1.1) with $G_{s,m,k,0}$ in place of $G_{s,m,k}$, where $s_h$ and $m_h$ are given by (1.3) and (1.4), respectively. We note that the first author [7] studied some asymptotic properties of sample paths near the boundaries of $I$ for ODGDP with generator $G_{s,m,0}$, and obtained a precise classification of the states of the boundaries by means of $s$, $m$, $k$ and $h$. We call $G_{s,m,0}$ the $h$ transform of $G_{s,m,k}$ with $h \in H_{s,m,k,\beta}$.

For $h \in \mathcal{M}_{s,m}$ [resp. $h \in H_{s,m,k,\beta}$], we set $H^u_{s,m}G_{s,m,0}u = h^{-1}G_{s,m,0}(hu)$ [resp. $H^u_{s,m,k+\beta}G_{s,m,0}u = h^{-1}G_{s,m,k+\beta}(hu)$] for $u$ satisfying $hu \in D(G_{s,m,0})$ [resp. $hu \in D(G_{s,m,k+\beta})$]. In the case that $G_{s,m,0}$ and $G_{s,m,k}$ are differential operators of second order, it is easy to see that $H^u_{s,m}G_{s,m,0}$ and $H^u_{s,m,k+\beta}G_{s,m,0}$ coincide with $G_{s,m,k,h}$ and $G_{s,m,k,h}$, respectively. We show that this is also true for ODGDOs.

**Theorem 1.2** Let $u$ be a measurable function on $I$.

(i) Let $h \in \mathcal{M}_{s,m}$. Then $u$ belongs to $D(G_{s,m,k,h})$ if and only if $hu$ belongs to $D(G_{s,m,0})$. Further $G_{s,m,k,h}u = H^u_{s,m}G_{s,m,0}u$ holds true for $u \in D(G_{s,m,k,h})$. 
(ii) Let \( h \in \mathcal{H}_{s,m,k,\beta} \). Then \( u \) belongs to \( D(\mathcal{G}_{s,m,k}) \) if and only if \( hu \) belongs to \( D(\mathcal{G}_{s,m,k}) \). Further \( \mathcal{G}_{s,m,k}u = H_h^* \mathcal{G}_{s,m,k+\beta}u \) holds true for \( u \in D(\mathcal{G}_{s,m,k}) \).

**Corollary 1.3** (i) Let \( h \in \mathcal{H}_{s,m,0,0} \). Then \( H_h^* \mathcal{G}_{s,m,0} = H_h^* \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0} \).

(ii) Let \( h \in \mathcal{M}_{s,m} \). Then \( H_h^* \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0} \) if and only if \( h \) is a positive constant function.

(iii) Let \( h \in \mathcal{H}_{s,m,k,\beta} \). Then \( H_h^* \mathcal{G}_{s,m,k+\beta} = \mathcal{G}_{s,m,k+\beta} \) if and only if \( k \) is the null measure, \( \beta = 0 \), and \( h \) is a positive constant function.

We next show that \( H_h^* \) and \( H_h^* \) are in inverse relation to each other in some sense.

**Theorem 1.4** (i) Let \( h \in \mathcal{M}_{s,m} \). Then \( h^{-1} \) belongs to \( \mathcal{H}_{s,m,h,k,0} \) and \( H_h^{-1} H_h^* \mathcal{G}_{s,m,0} = \mathcal{G}_{s,m,0} \).

(ii) Let \( h \in \mathcal{H}_{s,m,k,\beta} \). Then \( h^{-1} \) belongs to \( \mathcal{M}_{s,m,h} \) and \( H_h^{-1} H_h^* \mathcal{G}_{s,m,k+\beta} = \mathcal{G}_{s,m,k+\beta} \).

We should note that Theorem 1.4 does not necessarily ensure one-to-one correspondence of \( h \) transform. Indeed, some examples are given in Section 6, which show that \( h \) transform of ODGDO is not one-to-one correspondence.

Now Theorem 1.2 implies that an ODGDO is represented as an \( h \) transform of another ODGDO, but it does not always imply that an ODGDO is represented as an \( h \) transform of another ODGDO different from the original one. As we will see in the following theorem, this problem is related to a global property of ODGDO. We denote by \( \Phi \) the set of all ODGDOs on \( I \). Let \( \Phi^R \) be the set of all recurrent ODGDOs on \( I \), that is,

\[
\Phi^R = \{ \mathcal{G}_{s,m,k} \in \Phi : s(l_1) = -\infty, \ s(l_2) = \infty, \ \text{and} \ k \text{ is the null measure} \}.
\]

We set \( \Phi^T = \Phi \setminus \Phi^R \), which is the set of all transient ODGDOs on \( I \).

**Theorem 1.5** (i) Let \( \mathcal{G}_{s,m,0} \in \Phi^R \). If \( H_h^* \mathcal{G}_{s,m,0} \in \Phi^R \) for some \( h \in \mathcal{M}_{s,m} \), then \( h \) is a positive constant function. If \( H_h^* \mathcal{G}_{s,m,\beta} \in \Phi^R \) for some \( h \in \mathcal{H}_{s,m,0,\beta} \), then \( h \) is a positive constant function and \( \beta = 0 \).

(ii) Let \( \mathcal{G}_{s,m,0} \in \Phi^R \). If \( H_h^* \mathcal{G}_{s,m,0} \in \Phi^T \) for some \( h \in \mathcal{M}_{s,m} \), then \( h \) is not a positive constant function. If \( H_h^* \mathcal{G}_{s,m,\beta} \in \Phi^T \) for some \( h \in \mathcal{H}_{s,m,0,\beta} \), then \( h \) is not a positive constant function or \( \beta > 0 \). Conversely, if \( h \in \mathcal{M}_{s,m} \) and
is not a positive constant function, then $H_h G_{s,m,0} \in \Phi^T$. If $h \in \mathcal{H}_{s,m,0,\beta}$ and either $h$ is not a positive constant function or $\beta > 0$, then $H_h G_{s,m,\beta m} \in \Phi^T$.

(iii) If $G_{s,m,0} \in \Phi^T$, then $H_h G_{s,m,0} \in \Phi^T$ for any $h \in \mathcal{M}_{s,m}$. If $G_{s,m,k} \in \Phi^T$, then $H_h G_{s,m,k+\beta m} \in \Phi^T$ for any $h \in \mathcal{H}_{s,m,k,\beta}$.

We will show this theorem in Section 4.

We finally consider spectral representations of elementary solutions. We assume that supp$[m] = I$ and $l_1$ is not $(s,m,k)$-natural. Further we assume

\[ p(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda), \quad t > 0, \ x, y \in I, \quad (1.7) \]

where $d\sigma(\lambda)$ is a Borel measure on $[0, \infty)$ such that $\sigma(\{0\}) = 0$ if $l_1$ is $(s,m,k)$-regular or -exit. Further $\psi(x, \lambda), x \in I, \lambda \geq 0$, is the solution of the integral equation (5.1) or (5.2) below. Let $h \in \mathcal{H}_{s,m,k}$ and $p^*_h(t, x, y)$ be the elementary solution of the equation (1.1) with $G_{s_h, m_h, 0}$ in place of $G_{s,m,k}$, which is given by (1.6). By means of (1.6) and (1.7), we suppose $p^*_h(t, x, y)$ is represented as

\[ p^*_h(t, x, y) = \int_{[\beta, \infty)} e^{-\lambda t} \psi^*_h(x, \lambda) \psi^*_h(y, \lambda) d\sigma^*_h(\lambda), \quad t > 0, \ x, y \in I, \quad (1.8) \]

and $\psi^*_h(x, \lambda)$ and $d\sigma^*_h(\lambda)$ are represented as

\[ \psi^*_h(x, \lambda) = C_o \psi(x, \lambda - \beta)/h(x), \quad d\sigma^*_h(\lambda) = C_o^{-2} d\lambda \sigma(\lambda - \beta), \quad (1.9) \]

for $\lambda \geq \beta$, where $C_o$ is a positive constant function. In Section 5 we show that $\psi^*_h(x, \lambda)$ satisfies (5.1) or (5.2) with $(s,m,k)$ replaced by $(s_h, m_h, 0)$. Further we will find there that $C_o$ only depends on behavior of $s$, $m$, $k$ and $h$ near the boundary $l_1$, which leads us to an interesting behavior of Lévy measure density of inverse local time corresponding to a diffusion process on $I$ with the scale function $s_h$ and the speed measure $m_h$. We will discuss it in another paper.

The organization of this paper is as follows. In Section 2 we give the precise definitions of ODGDO $G_{s,m,k}$, the domain $D(G_{s,m,k})$ and the corresponding items. In Section 3 we give the precise definitions of $\mathcal{M}_{s,m}$ and $\mathcal{H}_{s,m,k,\beta}$, and prove Proposition 1.1 and Theorems 1.2, 1.4. In Section 4 we prove Theorem 1.5. In Section 5 we show that (1.8) holds true for $\psi^*_h(x, \lambda)$.
and \( d\sigma^*_i(\lambda) \) given by (1.9) with a suitable constant \( C_i \). It is easy to see that if \( H_{h_1}^* G_{s,m,0} = H_{h_2}^* G_{s,m,0} \) [resp. \( H_{h_1}^* G_{s,m,k+\beta m} = H_{h_2}^* G_{s,m,k+\beta m} \)] for some \( h_1, h_2 \in \mathcal{M}_{s,m} \) [resp. \( h_1, h_2 \in \mathcal{H}_{s,m,k,\beta} \)], then there is a positive constant \( K \) such that \( h_1 = K h_2 \). However it is not necessarily true that if \( H_{h_1}^* G_{s,m,0} = H_{h_2}^* G_{s,m,0} \) [resp. \( H_{h_1}^* G_{s,m,k+\beta m} = H_{h_2}^* G_{s,m,k+\beta m} \)] for some \( h_1 \in \mathcal{M}_{s,m} \) [resp. \( h_i \in \mathcal{H}_{s,m,k,\beta_i} \)] \((i = 1, 2)\), then \( h_1 = K h_2 \) for some positive constant \( K \) and \( G_{s,m,0} = G_{s,m,0} \) [resp. \( G_{s,m,k+\beta m} = G_{s,m,k+\beta m} \)]. We give such typical examples in Section 6.

2 Preliminaries

In this section we give the precise definitions of ODGDO \( D(G_{s,m,k}) \), the domain \( \mathcal{D}(G_{s,m,k}) \) and the corresponding items.

Let \( s \) be a continuous increasing function on an open interval \( I = (l_1, l_2) \), where \(-\infty \leq l_1 < l_2 \leq \infty \), \( m \) be a right continuous nondecreasing function on \( I \) and \( k \) be a right continuous nondecreasing function on \( I \). We sometimes use the same symbols \( s, m \) and \( k \) for the induced measures \( ds(x), dm(x) \) and \( dk(x) \), respectively. For a function \( u \) on \( I \), we set \( u(l_i) = \lim_{x \to l_i, x \in I} u(x) \) if there exists the limit, for \( i = 1, 2 \). We set

\[
I_*(\mu) = \{ x \in I; \mu(x_1) < \mu(x_2) \text{ for } \ell_1 < x_1 < x < x_2 < \ell_2 \},
\]

(2.1)

for a nondecreasing right continuous function \( \mu \) on \( I \). \( I_*(\mu) \) is the same as the support of the measure induced by \( \mu \). We assume \( I_*(m) \neq \emptyset \) and \( I_*(k) \subset I_*(m) \) throughout this paper. Further we set

\[
I_{ss}(m) = I_*(m) \cup \{ x; x = \ell_i \text{ with } |m(\ell_i)| + |s(\ell_i)| + |k(\ell_i)| < \infty, \ i = 1, 2 \}.
\]

Let us fix a point \( c_o \in I_*(m) \) arbitrarily and set

\[
J_{\mu,\nu}(x) = \int_{(c_o,x]} d\mu(y) \int_{(c_o,y]} d\nu(z),
\]

for \( x \in I \), where \( \mu \) and \( \nu \) are Borel measures on \( I \), and the integral \( \int_{(a,b]} \) is read as \(-\int_{[b,a]} \) if \( a > b \). Following [1], we call the boundary \( l_i \) to be

- \((s, m, k)\)-regular if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- \((s, m, k)\)-exit if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) = \infty \),
- \((s, m, k)\)-entrance if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- \((s, m, k)\)-natural if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) = \infty \).
Let \( D(G_{s,m,k}) \) be the space of all functions \( u \in L^2(I, m) \) which have a continuous version \( u \) (we use the same symbol) satisfying the following conditions:

i) There exist two constants \( A, B \) and a function \( f \in L^2(I, m) \) such that

\[
    u(x) = A + Bs(x) + \int_{(c_u,x]} \{s(x) - s(y)\} f_u(y) \, dm(y) \\
    + \int_{(c_u,x]} \{s(x) - s(y)\} u(y) \, dk(y), \quad x \in I.
\]

(2.2)

ii) If \( l_i \) is \((s, m, k)\)-regular, then \( u(l_i) = 0 \) for each \( i = 1, 2 \).

By virtue of (2.2), \( f_u \) is uniquely determined as a function of \( L^2(I, m) \) if it exists. The operator \( G_{s,m,k} \) from \( D(G_{s,m,k}) \) into \( L^2(I, m) \) is defined by

\[
    G_{s,m,k} u = f_u, 
\]

and it is called the one-dimensional generalized diffusion operator with the scale function \( s \), the speed measure \( m \), and the killing measure \( k \) (ODGDO with \((s, m, k)\) for brief). The above condition ii) implies that the absorbing boundary condition is posed at the regular boundary.

It is easy to see that \( G_{s_1, m_1, k_1} \) coincides with \( G_{s_2, m_2, k_2} \) if and only if there are a positive constant \( K \) and constants \( K_i, i = 1, 2, 3 \) such that \( s_1 = K s_2 + K_1, m_1 = K^{-1} m_2 + K_2, \) and \( k_1 = K^{-1} k_2 + K_3 \) (see [3]).

We note that \( l_i \) is \((s, m, k + \beta m)\)-regular, exit, entrance, or natural according to \( l_i \) is \((s, m, k)\)-regular, exit, entrance, or natural, for \( i = 1, 2 \) and \( \beta \geq 0 \). Combining this fact with (2.2), we immediately obtain the following.

\[
    D(G_{s,m,k}) = D(G_{s,m,k+\beta m}), \quad \text{(2.3)}
\]

\[
    G_{s,m,k} u - \beta u = G_{s,m,k+\beta m} u \quad \text{for} \quad u \in D(G_{s,m,k}), \quad \text{(2.4)}
\]

where \( \beta \geq 0 \).

In the following, for a measurable functions \( u \) on \( I \), \( D_s u(x) \) stands for the right derivative with respect to \( s(x) \), that is, \( D_s u(x) = \lim_{\varepsilon \to 0} \{u(x + \varepsilon) - u(x)\}/\{s(x + \varepsilon) - s(x)\} \), provided it exists. It is obvious that \( u \in D(G_{s,m,k}) \) has the right derivative \( D_s u \) and it satisfies

\[
    D_s u(y) - D_s u(x) = \int_{[x,y]} G_{s,m,k} u(z) \, dm(z) + \int_{[x,y]} u(z) \, dk(z), \quad x, y \in I.
\]

So we sometimes use the symbol \( G_{s,m,k} u = (dD_s u - u dk)/dm \).

Following McKean [5] (see also Section 4.11 of [3]), we can define the elementary solution \( p(t, x, y) \) of the equation (1.1). It is known that \( p(t, x, y) \)
satisfies the following properties.

\[ p(t, x, y) = p(t, y, x) > 0, \quad t > 0, \ x, y \in I. \]

\[ p(t, x, y) \text{ is continuous on } (0, \infty) \times I \times I. \]

\[ p(s + t, x, y) = \int_I p(s, x, z)p(t, z, y)\,dm(z), \quad s, t > 0, \ x, y \in I. \]

\[ p(t, l_i, y) = 0, \quad t > 0, \ y \in I, \quad \text{if } l_i \text{ is not entrance}. \]

\[ D_s p(t, l_i, y) = 0, \quad t > 0, \ y \in I, \quad \text{if } l_i \text{ is entrance}, \]

where \( D_s p(t, x, y) = \lim_{\varepsilon \downarrow 0} \{p(t, x + \varepsilon, y) - p(t, x, y)\} / \{s(x + \varepsilon) - s(x)\}. \) It is also known that there exists a one-dimensional generalized diffusion process (ODGDP for brief) \( \mathbb{D}_{s,m,k} = [X(t) : t \geq 0, \ P_x : x \in I^{**}(m)] \) such that

\[ P_x(X(t) \in E) = \int_E p(t, x, y)\,dm(y), \quad t > 0, \ x \in I^{**}(m), \ E \in \mathcal{B}(I), \]

where \( \mathcal{B}(I) \) stands for the set of all Borel sets of \( I. \) By this reason, \( p(t, x, y) \)

is sometimes called the transition probability density with respect to \( m. \)

For \( \alpha \geq 0 \) and \( i = 1, 2, \) let \( g_i(\cdot, \alpha) \) be a function on \( I \) satisfying the following properties (2.5)–(2.9).

\[ g_i(x, \alpha) \text{ is positive and continuous in } x. \quad (2.5) \]

\[ g_1(x, \alpha) \text{ is nondecreasing in } x. \quad (2.6) \]

\[ g_2(x, \alpha) \text{ is nonincreasing in } x. \quad (2.7) \]

\[ g_i(l_i, \alpha) = 0 \text{ if } |s(l_i)| < \infty. \quad (2.8) \]

\[ g_i(x, \alpha) = g_i(c_o, \alpha) + D_s g_i(c_o, \alpha)\{s(x) - s(c_o)\}

\[ + \int_{(c_o,x]} \{s(x) - s(y)\}g_i(y, \alpha)\{\alpha dm(y) + dk(y)\}, \quad x \in I. \quad (2.9) \]

Here \( D_s g_i(x, \alpha) = \lim_{\varepsilon \downarrow 0} \{g_i(x + \varepsilon, \alpha) - g_i(x, \alpha)\} / \{s(x + \varepsilon) - s(x)\}, \ i = 1, 2. \) It is known that there exist functions \( g_i(\cdot, \alpha), \ i = 1, 2, \) satisfying the properties (2.5)–(2.9) (see Section 4.6 of [3]). We summarize some properties of \( g_i(\cdot, \alpha), \ i = 1, 2, \) which we need later.
Proposition 2.1  (i) ([3]) Assume that $k$ is not a null measure or $\alpha > 0$. Then it holds true that
\[
g_i(l_i, \alpha) \begin{cases} 
\in (0, \infty) & \text{if } l_i \text{ is (s, m, k)-entrance}, \\
= 0 & \text{if } l_i \text{ is not (s, m, k)-entrance}; 
\end{cases}
\]
\[
g_j(l_i, \alpha) \begin{cases} 
\in (0, \infty) & \text{if } l_i \text{ is (s, m, k)-regular or exit}, \\
= \infty & \text{if } l_i \text{ is (s, m, k)-entrance or natural}; 
\end{cases}
\]
\[
|D_s g_i(l_i, \alpha)| \begin{cases} 
\in (0, \infty) & \text{if } l_i \text{ is (s, m, k)-regular or exit}, \\
= 0 & \text{if } l_i \text{ is (s, m, k)-entrance or natural}; 
\end{cases}
\]
\[
|D_s g_j(l_i, \alpha)| \begin{cases} 
\in (0, \infty) & \text{if } l_i \text{ is (s, m, k)-regular or entrance}, \\
= \infty & \text{if } l_i \text{ is (s, m, k)-exit or natural}; 
\end{cases}
\]
\[
\lim_{x \to l_i} g_i(x, \alpha)D_s g_j(x, \alpha) = 0 & \text{if } l_i \text{ is (s, m, k)-exit}, \\
\lim_{x \to l_i} g_j(x, \alpha)D_s g_i(x, \alpha) = 0 & \text{if } l_i \text{ is (s, m, k)-entrance};
\]

where $i, j = 1, 2$ and $i \neq j$.

(ii) Assume that $k$ is the null measure and $\alpha = 0$. Then $g_i(x, 0), i = 1, 2$ are represented as follows.

\[
g_1(x, 0) = \begin{cases} 
C_1 & \text{if } s(l_1) = -\infty \\
C_1 \{s(x) - s(l_1)\} & \text{if } s(l_1) > -\infty; 
\end{cases}
\]
\[
g_2(x, 0) = \begin{cases} 
C_2 & \text{if } s(l_2) = \infty \\
C_2 \{s(l_2) - s(x)\} & \text{if } s(l_2) < \infty; 
\end{cases}
\]

where $C_1$ and $C_2$ are positive constants.

The statement (i) is shown in Section 4.6 of [3]. The statement (ii) follows from (2.5) – (2.8). So we omit the proof.

We set $W(\alpha) = D_s g_1(x, \alpha) g_2(x, \alpha) - g_1(x, \alpha) D_s g_2(x, \alpha)$. Note that $W(\alpha)$ is a positive number independent of $x \in I$. We put
\[
G(\alpha, x, y) = G(\alpha, y, x) = W(\alpha)^{-1} g_1(x, \alpha) g_2(y, \alpha),
\]  
for $\alpha > 0$, $x, y \in I$, $x \leq y$. We call $G(\alpha, x, y)$ the $\alpha$-Green function corresponding to the ODGDO $G_{s,m,k}$. It is also known that
\[
G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) \, dt, \quad \alpha > 0, \; x, y \in I_s(m).
\]
It is easy to see that, if $k \neq 0$, then there exists $G(0, x, y)$ which is given by

$$G(0, x, y) = G(0, y, x) = W^{-1}g_1(x)g_2(y), \quad x, y \in I, \ x \leq y,$$

(2.12)

where $g_i(x) = g_i(x, 0)$, $i = 1, 2$, and $W = D_xg_1(x)g_2(x) - g_1(x)D_xg_2(x)$, which is a positive constant independent of $x \in I$. It follows from Proposition 2.1 that, in the case $k = 0$, there exists $G(0, x, y) \in (0, \infty)$ if and only if $|s(l_i)| < \infty$ for $i = 1$ or $2$.

We denote by $G_{\alpha}$ $(\alpha > 0)$ the Green operator corresponding to $G_{s,m,k}$.

$$G_{\alpha}f(x) = \int_I G(\alpha, x, y)f(y)\,dm(y), \quad f \in L^2(I, m).$$

(2.13)

It is well known that

$$G_{\alpha}(L^2(I, m)) = D(G_{s,m,k}),$$

(2.14)

$$G_{\alpha}(\alpha - G_{s,m,k})u = u, \quad u \in D(G_{s,m,k}),$$

(2.15)

$$(\alpha - G_{s,m,k})G_{\alpha}f = f, \quad f \in L^2(I, m),$$

(2.16)

(see [2] and [3]).

3 $h$ transform of ODGDOs

In this section, we give the precise definitions of $\mathcal{M}_{s,m}$ and $\mathcal{H}_{s,m,k,\beta}$, and prove Proposition 1.1 and Theorems 1.2, 1.4. We use the same notations as in the preceding sections.

3.1 $h$ transform of $G_{s,m,0}$ with $h \in \mathcal{M}_{s,m}$

Let $\mathcal{M}_{s,m}$ be the set of all positive continuous functions $h$ on $I$ such that $h$ has the right derivative $D_xh$, $D_xh$ is right continuous and nonincreasing, and the set \( \{ x \in I; D_xh(x_1) > D_xh(x_2) \text{ for } l_1 < \forall x_1 < x < \forall x_2 < l_2 \} \) is included in $I_*(m)$. For $h \in \mathcal{M}_{s,m}$, we consider $s_h$, $m_h$ and $k_h$ given by (1.3), (1.4) and (1.5), respectively. Note that $I_*(m) = I_*(m_h)$ and $I_*(k_h) \subset I_*(m_h)$ for $h \in \mathcal{M}_{s,m}$ (see [4]). Let $p^\alpha(t, x, y)$ be the elementary solution of (1.1) with $G_{s,m,k}$ replaced by $G_{s,m,0}$. Let $G^\alpha(\alpha, x, y)$ be the $\alpha$-Green function corresponding to the ODGDO $G_{s,m,0}$. For $h \in \mathcal{M}_{s,m}$, we consider $p^\alpha_h(t, x, y)$ given by (1.2) and set

$$G^\alpha_h(\alpha, x, y) = G^\alpha(\alpha, x, y)/h(x)h(y), \quad x, y \in I.$$  

(3.1)
The following result is obtained by Maeno (see Theorem 2.2 of [4]).

Proposition 3.1 ([4]) \( p^e_h(t, x, y) \) is the elementary solution of (1.1) with \( G_{s, m, k} \) replaced by \( G_{s_h, m_h, k_h} \), and \( G^*_h(\alpha, x, y) \) is the \( \alpha \)-Green function corresponding to the ODGDO \( G_{s_h, m_h, k_h} \).

3.2 \( h \) transform of \( G_{s, m, k} \) with \( h \in H_{s, m, k, \beta} \)

For \( \beta \geq 0 \), let \( h_\beta(\cdot) \) be a positive continuous function on \( I \) satisfying

\[
h_\beta(x) = h_\beta(c_o) + D_s h_\beta(c_o) \{s(x) - s(c_o)\} + \int_{(c_o, x]} \{s(x) - s(y)\} h_\beta(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.
\]

(3.2)

There exists such a function \( h_\beta(\cdot) \). Indeed, it is represented as a linear combination of \( g_i(\cdot, \beta) \), \( i = 1, 2 \), given in the preceding section. Let \( H_{s, m, k, \beta} \) be the set of all positive functions \( h_\beta \) satisfying (3.2). It immediately follows from (3.2) that

\[
H_{s, m, k, \beta} = H_{s, m, k + \beta m, 0}, \quad \beta \geq 0.
\]

For \( h \in H_{s, m, k, \beta} \) and \( g_i(x, \alpha) \), \( i = 1, 2 \), satisfying (2.5)–(2.9), we set

\[
g_{h,i}(x, \alpha) = g_i(x, \alpha) / h(x), \quad i = 1, 2, \quad \alpha \geq 0.
\]

(3.3)

Let \( G^*_h(\gamma, x, y) \) be the \( \gamma \)-Green function corresponding to \( G_{s_h, m_h, 0} \), where \( s_h \) and \( m_h \) are given by (1.3) and (1.4), respectively. Now we obtain the following proposition. Under the assumption \( I_*(m) = I \), the following result is obtained as Proposition 2.2 and Lemma 3.3 of [8]. It is not difficult to see that their proofs are available without the assumption \( I_*(m) = I \). Therefore we omit the proof of the following proposition.

Proposition 3.2 Let \( \alpha \geq \beta \geq 0 \) and \( h \in H_{s, m, k, \beta} \).

(i) Let \( i = 1, 2 \). If \( \alpha = \beta \) and \( |s_h(l_i)| = \infty \), then \( g_{h,i}(x, \alpha) \) is a positive constant function on \( I \). If \( \alpha > \beta \) or \( |s_h(l_i)| < \infty \), then \( g_{h,i}(x, \alpha) \) satisfies the following properties.

\[
\begin{align*}
(i-1) \quad & g_{h,i}(x, \alpha) \text{ is positive and continuous on } I. \\
(i-2) \quad & g_{h,1}(x, \alpha) \text{ is nondecreasing on } I \text{ and } g_{h,2}(x, \alpha) \text{ is nonincreasing on } I. \\
(i-3) \quad & \text{If } |s_h(l_i)| < \infty \text{ or } |m_h(l_i)| = \infty, \text{ then } g_{h,i}(l_i, \alpha) = 0. \\
(i-4) \quad & \text{If } |s_h(l_i)| = \infty, \text{ then } D_{s_h} g_{h,i}(l_i, \alpha) = 0.
\end{align*}
\]

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(i - 5) \( g_{h,i}(x, \alpha) \) satisfies
\[
g_{h,i}(x, \alpha) = g_{h,i}(c_o, \alpha) + D_s g_{h,i}(c_o, \alpha) \{ s_h(x) - s_h(c_o) \} + (\alpha - \beta) \int_{[c_o, x]} \{ s_h(x) - s_h(y) \} g_{h,i}(y, \alpha) \, dm_h(y), \tag{3.4}
\]
for \( x \in I \).

(ii) The following (3.5) holds true.
\[
G^*_h(\alpha - \beta, x, y) = G^*_h(\alpha - \beta, y, x) = W(\alpha)^{-1} g_{h,1}(x, \alpha) g_{h,2}(y, \alpha) = G(\alpha, x, y)/h(x)h(y), \tag{3.5}
\]
for \( l_1 < x \leq y < l_2 \).

Let \( p(t, x, y) \) be the elementary solution of the equation (1.1) and consider \( p^*_h(t, x, y) \) defined by (1.6). By virtue of (2.11) and (3.5), we get the following.

**Proposition 3.3** \( p^*_h(t, x, y) \) is the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( G_{s,h,m,0} \), and \( G^*_h(\alpha, x, y) \) is the \( \alpha \)-Green function corresponding to \( G_{s,h,m,0} \).

### 3.3 Proof of Proposition 1.1

(i) We set \( \Lambda = \{ h \in M_{s,m}; D_s h(x) \) is a constant function on \( I \} \). It follows from (3.2) that
\[
D_s h(y) - D_s h(x) = \int_{[x,y]} h(z) \{ \beta \, dm(z) + d k(z) \}, \tag{3.6}
\]
for \( h \in H_{s,m,k,\beta} \). Therefore, \( h \) belongs to \( H_{s,m,0,0} \) if and only if \( D_s h \) is a constant function, from which the set \( \{ x \in I; D_s h(x_1) > D_s h(x_2) \} \) for \( l_1 < \forall x_1 < x < \forall x_2 < l_2 \) is empty and hence it is included in \( I_*(m) \). Thus \( H_{s,m,0,0} \) is included in the set \( \Lambda \). Conversely, \( h \in \Lambda \) if and only if (3.6) holds true for the null killing measure and \( \beta = 0 \), which implies (3.2) with \( h = h_{\beta, k} \), \( k = 0 \) and \( \beta = 0 \). Therefore the set \( \Lambda \) is included in \( H_{s,m,0,0} \).

(ii) Assume that \( k \) is not a null measure or \( \beta > 0 \). Let \( h \in H_{s,m,k,\beta} \). By means of (3.6) we see that \( D_s h(x) < D_s h(y) \) whenever \( x < y \) and \( (x, y] \cap I_*(m) \neq \emptyset \). Therefore \( D_s h \) is increasing on \( I_*(m) \) and it is nondecreasing on \( I \). Thus \( h \) does not belong to \( M_{s,m} \), which implies the second statement. \( \square \)
3.4 Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. Let \( u \) be a measurable function on \( I \). We show the first statement (i). Let \( G_0^\alpha(\alpha, x, y) \) be the \( \alpha \)-Green function corresponding to \( G_{s,m,0} \). Let \( h \in \mathcal{M}_{s,m} \) and consider \( G_h^\alpha(\alpha, x, y) \) given by (3.1). By means of (2.14) and Proposition 3.1, \( u \) belongs to \( D(G_{s,h,m,k}^h) \) if and only if there is a function \( f \in L^2(I, m) \) such that

\[
    u(x) = \int_I G_h^\alpha(\alpha, x, y) f(y) \, dm(y), \quad (3.7)
\]

or equivalently,

\[
    u(x) = \frac{1}{h(x)} \int_I G_h^\alpha(\alpha, x, y) f(y) h(y) \, dm(y). \quad (3.8)
\]

We note that \( fh \) belongs to \( L^2(I, m) \). Therefore (3.8) holds true if and only if \( hu \) belongs to \( D(G_{s,m,0}) \). Proposition 3.1 combined with (3.7) and (3.8) implies

\[
    u(x) = G_h^\alpha(\alpha, x, y) f(x) = \frac{1}{h(x)} G_h^\alpha(fh)(x),
\]

where \( G_{h,\alpha}^\alpha \) and \( G_h^\alpha \) are the \( \alpha \)-Green operators corresponding to \( G_{s,h,m,k} \) and \( G_{s,m,0} \), respectively. By means of (2.16),

\[
    (\alpha - G_{s,h,m,k}^h) G_{h,\alpha}^\alpha f = f, \quad (\alpha - G_{s,m,0}) G_h^\alpha(fh) = fh,
\]

and hence we obtain

\[
    (\alpha - G_{s,h,m,k}) u = \frac{1}{h} (\alpha - G_{s,m,0})(hu),
\]

that is,

\[
    G_{s,h,m,k} u = \frac{1}{h} G_{s,m,0}(hu).
\]

Thus we get the statement (i).

The statement (ii) is obtained in the same way as above. So we omit the proof. \( \square \)

Proof of Corollary 1.3. The first statement (i) immediately follows from Proposition 1.1 and Theorem 1.2.
(ii) Let \( h \in M_{s,m} \). By means of Theorem 1.2, \( H^*_h G_{s,m,0} = G_{s,m,0} \) if and only if \( k_h \) is the null measure, and \( s_h(x) = A_s(x) + A_1 \), \( m_h(x) = A^{-1}m(x) + A_2 \) for some positive constant \( A \) and \( A_i \in \mathbb{R} \), \( i = 1, 2 \), or equivalently \( h \) is a positive constant function.

(iii) Let \( h \in H_{s,m,k,\beta} \). By means of Theorem 1.2, \( H^*_h G_{s,m,k+\beta m} = G_{s,m,k+\beta m} \) if and only if \( k + \beta m \) is the null measure, and \( s_h(x) = B_s(x) + B_1 \), \( m_h(x) = B^{-1}m(x) + B_2 \) for some positive constant \( B \) and \( B_i \in \mathbb{R} \), \( i = 1, 2 \). This is equivalent to \( k \) being the null measure, \( \beta = 0 \), and \( h \) being a positive constant function. \( \square \)

### 3.5 Proof of Theorem 1.4

(i) Let \( h \in M_{s,m} \). Then \( D_{s,h} h^{-1} = -D_s h \) and hence

\[
D_{s,h} h^{-1}(y) - D_{s,h} h^{-1}(x) = -D_s h(y) + D_s h(x)
\]

\[
= -\int_{(x,y]} dD_s h(z) = \int_{(x,y]} h^{-1}(z) dk_h(z).
\]

This shows that \( h^{-1} \) belongs to \( H_{s_h,m_h,k_h,0} \). By means of Theorem 1.2 (ii), \( H^*_h G_{s_h,m_h,k_h} = G_{s,m,0} \), from which \( H^*_h H^*_h G_{s,m,0} = G_{s,m,0} \).

(ii) Let \( h \in H_{s,m,k,\beta} \). Then \( D_{s,h} h^{-1} = -D_s h \), and by means of (3.6),

\[
D_{s,h} h^{-1}(y) - D_{s,h} h^{-1}(x) = -D_s h(y) + D_s h(x)
\]

\[
= -\int_{(x,y]} h(z) \{ \beta dm(z) + dk(z) \}.
\] (3.9)

Therefore \( D_{s,h} h^{-1} \) is nonincreasing on \( I \) and the set \( \Lambda = \{ x \in I : D_{s,h} h^{-1}(x_1) > D_{s,h} h^{-1}(x_2) \} \) for \( l_1 < \forall x_1 < x < \forall x_2 < l_2 \) is included in \( I_s(m) \cup I_s(k) \). Since \( I_s(m) \cup I_s(k) = I_s(m) = I_s(m_\beta) \), we get \( h^{-1} \in M_{s_h,m_h} \). Calculating the right hand side of (1.5) with \( s \) and \( h \) replaced by \( s_h \) and \( h^{-1} \), respectively, we see by virtue of (3.9) that

\[
-\int_{(c,x]} h^{-1}(y) dD_s h^{-1}(y) = \int_{(c,x]} \{ \beta dm(y) + dk(y) \}.
\]

Combining this with Theorem 1.2 (i), we get \( H^*_h G_{s_h,m_h,0} = G_{s,m,k+\beta m} \), from which \( H^*_h H^*_h G_{s,m,k+\beta m} = G_{s,m,k+\beta m} \). \( \square \)
4 Global properties of ODGDOs and their $h$ transform

In this section we show Theorem 1.5. First we prepare two lemmas on properties of $h \in \mathcal{M}_{s,m}$ and $h \in \mathcal{H}_{s,m,k,\beta}$.

**Lemma 4.1** Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$. If $h \in \mathcal{M}_{s,m}$ and $k_h$ is the null measure, then $h$ is a positive constant function. If $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$, then $h$ is a positive constant function and $\beta = 0$.

**Proof.** Assume that $s(l_1) = -\infty$ and $s(l_2) = \infty$.

(i) Suppose that $h \in \mathcal{M}_{s,m}$ and $k_h$ is the null measure. By means of (1.5), $D_s h$ is a constant function. Therefore $h(x)$ is represented as $h(x) = C_1 s(x) + C_2$ for some $C_i \in \mathbb{R}$, $i = 1, 2$. Since $s(l_1) = -\infty$, $s(l_2) = \infty$ and $h(x)$ is a positive function, we get $C_1 = 0$ and $C_2 > 0$.

(ii) Suppose that $h \in \mathcal{H}_{s,m,0,\beta}$, $s_h(l_1) = -\infty$ and $s_h(l_2) = \infty$. By using Lemma 3.2 (i) of [8], we see that $h(l_i) < \infty$, $i = 1, 2$. Note that $h(x)$ is represented as $h(x) = C_3 g_1(x, \beta) + C_4 g_2(x, \beta)$, where $g_i(x, \beta)$, $i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$ and $k = 0$. If $\beta > 0$, by means of Proposition 2.1 (i), $g_i(l_i, \beta) < \infty$ for $i = 1, 2$, and $g_i(l_j, \beta) = \infty$ for $i \neq j$. Combining these with $h(l_i) < \infty$ for $i = 1, 2$, we see $C_3 = C_4 = 0$, and hence $h(x) = 0$. This contradicts $h(x) > 0$ on $I$. Thus $\beta = 0$. Noting $s(l_1) = -\infty$, $s(l_2) = \infty$ and Proposition 2.1 (ii), we obtain that $h$ is a positive constant function. \hfill $\square$

**Lemma 4.2**

(i) Let $h \in \mathcal{M}_{s,m}$. If $s(l_1) > -\infty$ or $s(l_2) < \infty$, then $s_h(l_1) > -\infty$, or $s_h(l_2) < \infty$, or $k_h$ is not a null measure.

(ii) Let $h \in \mathcal{H}_{s,m,k,\beta}$. If $s(l_1) > -\infty$, or $s(l_2) < \infty$, or $k$ is not a null measure, then $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$.

**Proof.** (i) Let $h \in \mathcal{M}_{s,m}$, and $s(l_1) > -\infty$ or $s(l_2) < \infty$. It is enough to show that $s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$ in the case that $k_h$ is the null measure. Since $k_h$ is the null measure, $D_s h$ is a constant function by virtue of (1.5).

Suppose $s(l_1) > -\infty$. By means of Lemma 2.1 (ii) of [4], we have $h(l_1) \in [0, \infty)$. If $h(l_1) \in (0, \infty)$, then $s_h(l_1) > -\infty$. Let $h(l_1) = 0$. Noting that $D_s h$ is a constant function, we see that $h(x) = C_1 \{s(x) - s(l_1)\}$ for some $C_1 > 0$. (1.3) coupled with this implies $s_h(l_2) < \infty$.  

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In the same way as above, we get \( s_h(l_1) > -\infty \) or \( s_h(l_2) < \infty \) when \( s(l_2) < \infty \).

(ii) Let \( h \in \mathcal{H}_{s,m,k,\beta} \) and assume that \( s(l_1) > -\infty \), or \( s(l_2) < \infty \), or \( k \) is not a null measure. Note \( h(x) \) is represented as \( h(x) = C_2g_1(x, \beta) + C_3g_2(x, \beta) \), where \( g_i(x, \beta) \), \( i = 1, 2 \), are functions satisfying (2.5)–(2.9) with \( \alpha = \beta \). We divide the proof into four cases.

Case 1: \( h(l_1) = \infty \). Then \( s_h(l_1) > -\infty \) by virtue of Lemma 3.2 (i) of [8].

Case 2: \( h(l_1) \in (0, \infty) \) and \( s(l_1) > -\infty \). Then \( s_h(l_1) > -\infty \) by means of (1.3).

Case 3: \( h(l_1) \in (0, \infty) \) with \( s(l_1) = -\infty \). Since \( g_2(l_1, \beta) = \infty \) for \( \beta \geq 0 \) by virtue of Proposition 2.1, we get \( C_3 = 0 \) and \( h(x) = C_2g_1(x, \beta) \).

(3-1) Let \( s(l_2) < \infty \). Then

\[
sh(l_2) = \int_{(c_o,l_2)} h(y)^{-2} ds(y) \leq C_2^{-2}g_1(c_o, \beta)^{-2}\{s(l_2) - s(c_o)\} < \infty.
\]

(3-2) Let \( s(l_2) = \infty \). Since \( k \) is not a null measure, we have, by virtue of Proposition 2.1 (i), \( h(l_2) = C_2g_1(l_2, \beta) = \infty \) for \( \beta \geq 0 \). Therefore \( s_h(l_2) < \infty \) by virtue of Lemma 3.2 (i) of [8].

Case 4: \( h(l_1) = 0 \). Then \( h(x) \) is represented as \( h(x) = C_2g_1(x, \beta) \), because of \( g_2(l_1, \beta) > 0 \). Thus we have \( s_h(l_2) < \infty \) in the same way as in (3-1) and (3-2).

\[\square\]

**Proof of Theorem 1.5.** (i) Let \( \mathcal{G}_{s,m,0} \in \Phi^R \). Hence \( s(l_1) = -\infty \) and \( s(l_2) = \infty \).

Suppose that \( H_h^0\mathcal{G}_{s,m,0} = \mathcal{G}_{s_h, m_h, k_h} \in \Phi^R \) for some \( h \in \mathcal{M}_{s,m} \). Then \( k_h \) is the null measure. By virtue of Lemma 4.1, \( h \) is a positive constant function.

Suppose that \( H_h^0\mathcal{G}_{s,m,\beta m} = \mathcal{G}_{s_h, m_h, \beta m} \in \Phi^R \) for some \( h \in \mathcal{H}_{s,m,0,\beta} \). Then \( s_h(l_1) = -\infty \) and \( s_h(l_2) = \infty \). By virtue of Lemma 4.1, \( h \) is a positive constant function and \( \beta = 0 \).

(ii) The former statement follows from Corollary 1.3. The later follows from Lemma 4.1.

(iii) (1) Let \( \mathcal{G}_{s,m,0} \in \Phi^T \) and \( h \in \mathcal{M}_{s,m} \). Since \( s(l_1) > -\infty \) or \( s(l_2) < \infty \), by means of Lemma 4.2 (i), \( s_h(l_1) > -\infty \) or \( s_h(l_2) < \infty \) or \( k_h \) is not a null measure. Thus \( H_h^0\mathcal{G}_{s,m,0} = \mathcal{G}_{s_h, m_h, k_h} \in \Phi^T \).

(2) Let \( \mathcal{G}_{s,m,k} \in \Phi^T \) and \( h \in \mathcal{H}_{s,m,k,\beta} \). Noting that \( s(l_1) > -\infty \), or \( s(l_2) < \infty \), or \( k \) is not a null measure, and using Lemma 4.2 (ii), we see
$s_h(l_1) > -\infty$ or $s_h(l_2) < \infty$. This implies $H_n^* g_{s,m,k} = g_{s,m,k} \in \Phi^T$.

5 Spectral representation of elementary solution

Let $G_{s,m,k}$ be an ODGDO and $p(t,x,y)$ be the elementary solution of the equation (1.1). We assume that $l_1 = I$ and $l_1$ is not $(s,m,k)$-natural. Further we assume that $p(t,x,y)$ is represented as (1.7). Namely,

$$p(t,x,y) = \int_{[0,\infty)} e^{-\lambda t} \psi(x,\lambda) \psi(y,\lambda) d\sigma(\lambda), \quad t > 0, \ x, y \in I,$$

where $d\sigma(\lambda)$ is a Borel measure on $[0,\infty)$; $\sigma(\{0\}) = 0$ if $l_1$ is $(s,m,k)$-regular or exit. Further $\psi(x,\lambda), \ x \in I, \ \lambda \geq 0$, is the solution of the following integral equation.

$$\psi(x,\lambda) = s(x) - s(l_1) + \int_{[l_1,x]} \{s(x) - s(y)\} \psi(y,\lambda) \{ -\lambda dm(y) + dk(y) \},$$

if $l_1$ is $(s,m,k)$-regular or exit. (5.1)

$$\psi(x,\lambda) = 1 + \int_{[l_1,x]} \{s(x) - s(y)\} \psi(y,\lambda) \{ -\lambda dm(y) + dk(y) \},$$

if $l_1$ is $(s,m,k)$-entrance. (5.2)

In the following we fix an $h \in H_{s,m,k}$ arbitrarily. Then $G_{s,m,k}$ is an ODGDO and $p_h(t,x,y)$ given by (1.6) is the elementary solution of the equation (1.1) with $G_{s,m,k}$ in place of $G_{s,m,k}$. We set $\psi_h(x,\lambda) = \psi(x,\lambda)/h(x)$. By virtue of (1.6) and (1.7), we obtain

$$p_h(t,x,y) = e^{-\beta t} p_h(t,x,y) = \int_{[0,\infty)} e^{-\lambda t} \psi_h(x,\lambda - \beta) \psi_h(y,\lambda - \beta) d\sigma(\lambda - \beta).$$

In the same way as in the proof of Lemma 3.3 of [8], we see that $\psi_h(x,\lambda)$ satisfies the following.

$$D_{s_h} \psi_h(y,\lambda) - D_{s_h} \psi_h(x,\lambda) = -(\lambda + \beta) \int_{[x,y]} \psi_h(z,\lambda) dm_h(z),$$

(5.4)
Lemma 5.1  

(i) Let $l_1$ be $(s, m, k)$-regular or exit. Then $h(l_1) \in [0, \infty)$ and

\[
\psi_h(l_1, \lambda) = D_s h(l_1)^{-1} \in (0, \infty), \quad D_{s_h} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) = 0; \quad (5.5)
\]

\[
\psi_h(l_1, \lambda) = 0, \quad D_{s_h} \psi_h(l_1, \lambda) = h(l_1) \in (0, \infty), \quad \text{if } h(l_1) \in (0, \infty). \quad (5.6)
\]

(ii) Let $l_1$ be $(s, m, k)$-entrance. Then $h(l_1) \in (0, \infty]$ and

\[
\psi_h(l_1, \lambda) = h(l_1)^{-1} \in (0, \infty), \quad D_{s_h} \psi_h(l_1, \lambda) = 0, \quad \text{if } h(l_1) \in (0, \infty); \quad (5.7)
\]

\[
\psi_h(l_1, \lambda) = 0, \quad D_{s_h} \psi_h(l_1, \lambda) = -D_s h(l_1) \in (0, \infty), \quad \text{if } h(l_1) = \infty. \quad (5.8)
\]

Proof. Note that $h$ is represented as $h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)$, where $g_i(x, \beta), i = 1, 2$, are functions satisfying (2.5)–(2.9) with $\alpha = \beta$.

(i) Let $l_1$ be $(s, m, k)$-regular or exit. By means of Theorem 1.1 of [7], $0 \leq h(l_1) < \infty$.

First we consider the case $h(l_1) = 0$, and hence $C_1 > 0$ and $C_2 = 0$, by virtue of Proposition 2.1. Since $D_s h(l_1) = C_1 D_s g_1(l_1, \beta) \in (0, \infty)$, we get

\[
\psi_h(l_1, \lambda) = \lim_{x \to l_1^-} \frac{\psi(x, \lambda)}{h(x)} = \lim_{x \to l_1^-} \frac{D_s \psi(x, \lambda)}{D_s h(x)} = \frac{1}{D_s h(l_1)} \in (0, \infty),
\]

\[
D_{s_h} \psi_h(l_1, \lambda) = \lim_{x \to l_1^-} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = 0.
\]

Thus we obtain (5.5).

Next we consider the case $0 < h(l_1) < \infty$. Then $C_2 > 0$ and $\psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0$. If $l_1$ is $(s, m, k)$-regular, then by means of Proposition 2.1, $D_s h(l_1) = C_1 D_s g_1(l_1, \beta) + C_2 D_s g_2(l_1, \beta) \in \mathbb{R}$ and hence

\[
D_{s_h} \psi_h(l_1, \lambda) = \lim_{x \to l_1^-} \{h(x) D_s \psi(x, \lambda) - \psi(x, \lambda) D_s h(x)\} = h(l_1) \in (0, \infty).
\]

If $l_1$ is $(s, m, k)$-exit, then by means of Proposition 2.1

\[
\lim_{x \to l_1^-} \frac{\psi(x, \lambda)}{g_1(x, \beta)} = \lim_{x \to l_1^-} \frac{D_s \psi(x, \lambda)}{D_s g_1(x, \beta)} = \frac{1}{D_s g_1(l_1, \beta)} \in (0, \infty),
\]

\[
\lim_{x \to l_1^-} \psi(x, \lambda) D_s h(x)
\]

\[
= \frac{1}{D_s g_1(l_1, \beta)} \lim_{x \to l_1^-} g_1(x, \beta) \{C_1 D_s g_1(x, \beta) + C_2 D_s g_2(x, \beta)\} = 0.
\]
Therefore we arrive at
\[ D_{sh} \psi_h(l_1, \lambda) = \lim_{x \to l_1} \{h(x)D_s \psi(x, \lambda) - \psi(x, \lambda)D_s h(x)\} \]
\[ = h(l_1) \in (0, \infty). \]

Thus we have (5.6).

(ii) Let \( l_1 \) be \((s, m, k)\)-entrance. By means of Theorem 1.1 of [7], \( 0 < h(l_1) \leq \infty \).

First we consider the case \( 0 < h(l_1) < \infty \), and hence \( C_1 > 0 \) and \( C_2 = 0 \) by virtue of Proposition 2.1. Then
\[ \psi_h(l_1, \lambda) = \frac{1}{h(l_1)} = \frac{1}{C_1 g_1(l_1, \beta)} \in (0, \infty), \]
\[ D_{sh} \psi_h(l_1, \lambda) = h(l_1)D_s \psi(l_1, \lambda) - \psi(l_1, \lambda)D_s h(l_1) = -C_1 D_s g_1(l_1, \beta) = 0. \]

These show (5.7).

Next we consider the case \( h(l_1) = \infty \). Then \( C_2 > 0 \) and \( \psi_h(l_1, \lambda) = \psi(l_1, \lambda)/h(l_1) = 0. \) We show that
\[ D_{sh} \psi_h(l_1, \lambda) = -D_s h(l_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty). \] (5.9)

Here is the proof of (5.9). Since \( l_1 \) is \((s, m, k)\)-entrance, for any positive \( \varepsilon \) there exists an \( r \in I \) such that
\[ \{s(r) - s(x)\} \int_{[l_1, x]} (\lambda \, dm(z) + dk(z)) \]
\[ \leq \int_{[l_1, x]} \{s(r) - s(z)\}(\lambda \, dm(z) + dk(z)) \]
\[ \leq \int_{[l_1, r]} \{s(r) - s(z)\}(\lambda \, dm(z) + dk(z)) < \varepsilon, \] (5.10)
for \( l_1 < x < r \). It is easy to derive the following estimate from (5.2).
\[ |D_s \psi(x, \lambda)| \leq \int_{[l_1, x]} (\lambda \, dm(z) + dk(z)) \exp \left\{ \int_{[l_1, x]} (\lambda \, dm(z) + dk(z)) \int_{(z, x]} ds(y) \right\}. \]

Combining this with (5.10), we find
\[ \limsup_{x \to {l_1}} \{s(r) - s(x)\}|D_s \psi(x, \lambda)| < \varepsilon. \]
Therefore

\[
\limsup_{x \to t_1} h(x)|D_s \psi(x, \lambda)| \leq \varepsilon \lim_{x \to t_1} \frac{h(x)}{s(r) - s(x)} = -\varepsilon D_s h(t_1) = -\varepsilon C_2 D_s g_2(\lambda_1, \beta).
\]

Since \( D_s g_2(l_1, \beta) \in (-\infty, 0) \) by virtue of Proposition 2.1, letting \( \varepsilon \downarrow 0 \) leads us to \( \lim_{x \to t_1} h(x)D_s \psi(x, \lambda) = 0 \), and

\[
D_s h(t_1, \lambda) = \lim_{x \to t_1} \{h(x)D_s \psi(x, \lambda) - \psi(x, \lambda)D_s h(x)\} = -D_s h(t_1) = -C_2 D_s g_2(l_1, \beta) \in (0, \infty),
\]

which shows (5.9). Thus we obtain (5.8). \( \square \)

Now we define \( \psi^*_h(x, \lambda), x \in I, \lambda \geq \beta \), and \( d\sigma^*_h(\lambda), \lambda \geq \beta \), as follows. 

Case 1. \( l_1 \) is \((s, m, k)\)-regular or exit. If \( h(l_1) = 0 \), then

\[
\psi^*_h(x, \lambda) = D_s h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \tag{5.11}
\]

\[
d\sigma^*_h(\lambda) = \{D_s h(l_1)\}^{-2} d_\lambda \sigma(\lambda - \beta). \tag{5.12}
\]

If \( h(l_1) \in (0, \infty) \), then

\[
\psi^*_h(x, \lambda) = \{h(l_1) h(x)\}^{-1} \psi(x, \lambda - \beta), \tag{5.13}
\]

\[
d\sigma^*_h(\lambda) = h(l_1)^2 d_\lambda \sigma(\lambda - \beta). \tag{5.14}
\]

Case 2. \( l_1 \) is \((s, m, k)\)-entrance. If \( h(l_1) \in (0, \infty) \), then

\[
\psi^*_h(x, \lambda) = h(l_1) h(x)^{-1} \psi(x, \lambda - \beta), \tag{5.15}
\]

\[
d\sigma^*_h(\lambda) = h(l_1)^{-2} d_\lambda \sigma(\lambda - \beta). \tag{5.16}
\]

If \( h(l_1) = \infty \), then

\[
\psi^*_h(x, \lambda) = \{-D_s h(l_1) h(x)\}^{-1} \psi(x, \lambda - \beta), \tag{5.17}
\]

\[
d\sigma^*_h(\lambda) = \{D_s h(l_1)\}^2 d_\lambda \sigma(\lambda - \beta). \tag{5.18}
\]

We note that \( \sigma^*_h(\{\beta\}) = 0 \) in Case 1, but \( \sigma^*_h(\{\beta\}) \geq 0 \) in Case 2.

By means of Theorem 1.1 of [7], \( l_1 \) is not \((s_h, m_h, 0)\)-natural. More precisely, in the case \( l_1 \) is \((s, m, k)\)-regular [resp. exit],

- \( l_1 \) is \((s_h, m_h, 0)\)-regular [resp. exit] if \( h(l_1) \in (0, \infty) \),
- \( l_1 \) is \((s_h, m_h, 0)\)-entrance if \( h(l_1) = 0 \).

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in the case \(l_1\) is \((s_h, m_h, 0)\)-entrance,

\[
l_1 \text{ is } (s_h, m_h, 0)\text{-entrance} \quad \text{if } h(l_1) \in (0, \infty),
\]

\[
l_1 \text{ is } (s_h, m_h, 0)\text{-regular} \quad \text{if } h(l_1) = \infty \text{ and } |m_h(l_1)| < \infty,
\]

\[
l_1 \text{ is } (s_h, m_h, 0)\text{-exit} \quad \text{if } h(l_1) = \infty \text{ and } |m_h(l_1)| = \infty.
\]

Therefore (5.3), (5.4) and Lemma 5.1 lead us to the following result.

**Proposition 5.2** \(\psi^*_h(x, \lambda)\) satisfies (5.1) or (5.2) with \(s, m\) and \(k\) replaced by \(s_h, m_h\) and \(0\), respectively. \(d\sigma^*_h(\lambda)\) is a Borel measure on \([\beta, \infty)\). \(p_h^*(t, x, y)\) is represented as

\[
p_h^*(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi^*_h(x, \lambda) \psi^*_h(y, \lambda) d\sigma^*_h(\lambda), \quad t > 0, \ x, y \in I. \quad (5.19)
\]

**6 Examples**

**Example 6.1** First we consider the following ODGDO \(G_{s,m,k}\) on \(I = (0, \infty)\) with constant coefficients.

\[
G_{s,m,k} = a \frac{d^2}{dx^2} + b \frac{d}{dx} - c, \quad (6.1)
\]

where \(a > 0, \ b \in \mathbb{R}\) and \(c \geq 0\). We may set

\[
ds(x) = e^{-(b/a)x} dx, \ dm(x) = a^{-1} e^{(b/a)x} dx, \ dk(x) = (c/a) e^{(b/a)x} dx. \quad (6.2)
\]

The end point 0 is \((s, m, k)\)-regular and the end point \(\infty\) is \((s, m, k)\)-natural. For \(\alpha \geq 0\), we set

\[
\lambda_i(\alpha) = \frac{1}{2a} \left\{ (-1)^i b + \sqrt{b^2 + 4a(c + \alpha)} \right\}, \quad i = 1, 2.
\]

The \(\alpha\)-Green function \(G(\alpha, x, y)\) corresponding to \(G_{s,m,k}\) is given by the following.

\[
G(\alpha, x, y) = G(\alpha, y, x)
\]

\[
= \begin{cases} 
(\lambda_1(\alpha) + \lambda_2(\alpha))^{-1} g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha > 0, \\
g_1(x, \alpha) g_2(y, \alpha), & \text{if } b^2 + c + \alpha = 0,
\end{cases}
\]

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for $0 \leq x \leq y < \infty$, where

\[
g_1(x, \alpha) = \begin{cases} 
e^{\lambda_1(\alpha)x} - e^{-\lambda_2(\alpha)x}, & \text{if } b^2 + c + \alpha > 0, \\ x, & \text{if } b^2 + c + \alpha = 0, \end{cases}
\]

\[
g_2(x, \alpha) = \begin{cases} e^{-\lambda_2(\alpha)x}, & \text{if } b^2 + c + \alpha > 0, \\ 1, & \text{if } b^2 + c + \alpha = 0. \end{cases}
\]

The elementary solution $p(t, x, y)$ is given by

\[
p(t, x, y) = \frac{1}{2} \sqrt{\frac{a}{\pi t}} \exp \left\{ -\frac{b}{2a}(x + y) - At \right\} \left\{ e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right\}
\]

\[
= \int_{A}^{\infty} e^{-\lambda t} \psi(x, \lambda) \psi(y, \lambda) d\sigma(\lambda),
\]

where $A = b^2/4a + c$ and

\[
\psi(x, \lambda) = e^{-(b/2a)x} \frac{a}{\lambda - A} \sin \left( \sqrt{\frac{\lambda - A}{a}} x \right),
\]

\[
d\sigma(\lambda) = \frac{1}{\pi} \sqrt{\frac{\lambda - A}{a}} d\lambda,
\]

for $x \in I$ and $\lambda > A$.

Let $h \in \mathcal{H}_{s,m,k,\beta}$. Then the $h$ transform of $G_{s,m,k}$ is reduced to the following.

\[
G_{ss,ms,0} = a \frac{d^2}{dx^2} + \left( b + 2a \frac{h'(x)}{h(x)} \right) \frac{d}{dx}.
\]  

(1) Assume $h(0) \in (0, \infty)$. Then $h$ is represented as $h(x) = C_1 g_1(x, \beta) + C_2 g_2(x, \beta)$ for some constants $C_1 \geq 0$ and $C_2 > 0$. The end point 0 is $(s_h, m_h, 0)$-regular and we get (5.19), where by means of (5.13) and (5.14),

\[
\psi^*_h(x, \lambda) = \{C_2 h(x)\}^{-1} \psi(x, \lambda - \beta), \quad d\sigma^*_h(\lambda) = C_2^2 d\lambda \sigma(\lambda - \beta).
\]  

(2) Assume $h(0) = 0$. Then $h(x) = C_1 g_1(x, \beta)$ with $C_1 > 0$. The end point 0 is $(s_h, m_h, 0)$-entrance and we get (5.19), where by means of (5.11) and (5.12),

\[
\psi^*_h(x, \lambda) = \tilde{C} h(x)^{-1} \psi(x, \lambda - \beta), \quad d\sigma^*_h(\lambda) = \tilde{C}^{-2} d\lambda \sigma(\lambda - \beta).
\]
Here
\[ \tilde{C} = D_s h(0) = \begin{cases} C_1(\lambda_1(\beta) + \lambda_2(\beta)), & \text{if } b^2 + c + \beta > 0, \\ C_1, & \text{if } b^2 + c + \beta = 0. \end{cases} \]

Especially, if \( b = c = 0 \) and \( C_1 = 1 \), then \( \lambda_i(\beta) = \sqrt{\beta/a}, \ i = 1, 2, \) and
\[ h(x) = \begin{cases} 2 \sinh(\sqrt{\beta/a} x), & \text{if } \beta > 0, \\ x, & \text{if } \beta = 0, \end{cases} \]
\[ \tilde{C} = \begin{cases} 2\sqrt{\beta/a}, & \text{if } \beta > 0, \\ 1, & \text{if } \beta = 0. \end{cases} \]

Therefore (6.3) and (5.19) are reduced to the following. If \( \beta > 0 \), then
\[ G_{s_h, n, 0} = a \frac{d^2}{dx^2} + 2\sqrt{a\beta} \frac{\cosh(\sqrt{\beta/a} x)}{\sinh(\sqrt{\beta/a} x)} \frac{d}{dx}, \]
\[ p_h^*(t, x, y) = \frac{1}{8} \sqrt{\frac{a}{\pi t}} e^{-\beta t} \left\{ e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right\} / \sinh(\sqrt{\beta/a} x) \sinh(\sqrt{\beta/a} y), \]
\[ = \int_\beta^\infty e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \]
where
\[ \psi_h^*(x, \lambda) = \sqrt{\frac{\beta}{\lambda - \beta}} \frac{\sin(\sqrt{(\lambda - \beta)/a} x)}{\sinh(\sqrt{\beta/a} x)}, \quad d\sigma_h^*(\lambda) = \frac{1}{4\pi} \sqrt{\frac{(\lambda - \beta)/\beta}{\sqrt{\beta/a}}} d\lambda. \]

If \( \beta = 0 \), then
\[ G_{s_h, n, 0} = a \frac{d^2}{dx^2} + \frac{2a}{x} \frac{d}{dx}, \quad (6.5) \]
\[ p_h^*(t, x, y) = \frac{1}{2} \sqrt{\frac{a}{\pi t xy}} \left\{ e^{-(x-y)^2/4at} - e^{-(x+y)^2/4at} \right\} \]
\[ = \int_0^\infty e^{-\lambda t} \psi_h^*(x, \lambda) \psi_h^*(y, \lambda) d\sigma_h^*(\lambda), \quad (6.6) \]
where
\[ \psi_h^*(x, \lambda) = \frac{\sin(\sqrt{\beta/a} x)}{\sqrt{\lambda/a} x}, \quad d\sigma_h^*(\lambda) = \frac{1}{\pi} \sqrt{\frac{\lambda}{a}} d\lambda. \]

When \( a = 1/2 \), (6.5) is the generator of 3-dimensional Bessel process which we denote by \( \tilde{G} \), that is,
\[ \tilde{G} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx}. \]
Since \( \tilde{h}(x) = 1/x \) satisfies \( \tilde{G}\tilde{h} = 0 \), we can consider the \( h \) transform of \( \tilde{G} \) with respect to \( \tilde{h}(x) = 1/x \), which we denote by \( G \). Then we get

\[
G = \frac{1}{2} \frac{d^2}{dx^2}.
\]

This is the generator of Brownian motion on \((0, \infty)\). The above observation shows that \( G \) is \( h \) transformed to \( \tilde{G} \) in terms of \( h(x) = x \), and \( \tilde{G} \) is \( h \) transformed to \( G \) in terms of \( \tilde{h}(x) = 1/x \). This fact is also obtained by means of Theorem 1.4.

(3) We denote \( G_{s,m,k} \) with \( b^2 + c > 0 \) [resp. \( b^2 + c = 0 \)] by \( G^{(1)} \) [resp. \( G^{(2)} \)]. Fix an \( \beta \geq 0 \) arbitrarily, and set \( h^{(1)}(x) = e^{\lambda_1(\beta)x} \). Since \( G^{(1)}h^{(1)} = \kappa h^{(1)} \), we can consider an \( h \) transform of \( G^{(1)} \) with respect to \( h^{(1)} \) which is denoted by \( \tilde{G}^{(1)} \), that is,

\[
\tilde{G}^{(1)} = a \frac{d^2}{dx^2} + \left( b + 2a \frac{h^{(1)'}(x)}{h^{(1)}(x)} \right) \frac{d}{dx} = a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}.
\] (6.7)

We put \( \kappa = b^2/4a + c + \beta \) and \( h^{(2)}(x) = e^{\sqrt{\kappa/a}x} \). Since \( G^{(2)}h^{(2)} = \kappa h^{(2)} \), we can consider an \( h \) transform of \( G^{(2)} \) with respect to \( h^{(2)} \) which is denoted by \( \tilde{G}^{(2)} \), that is,

\[
\tilde{G}^{(2)} = a \frac{d^2}{dx^2} + 2a \frac{h^{(2)'}(x)}{h^{(2)}(x)} \frac{d}{dx} = a \frac{d^2}{dx^2} + 2a \sqrt{\kappa/a} \frac{d}{dx} = a \frac{d^2}{dx^2} + \sqrt{b^2 + 4a(c + \beta)} \frac{d}{dx}.
\] (6.8)

We find that \( G^{(1)} \neq G^{(2)} \), but \( \tilde{G}^{(1)} = \tilde{G}^{(2)} \). This shows that \( h \) transform of ODGDO is not one-to-one correspondence.

**Example 6.2** Let us consider the following ODGDO \( G_\kappa \) on \( I = (0, \infty) \).

\[
G_\kappa = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\kappa}{2x} \frac{d}{dx}.
\]
where $\kappa$ is a real number. $G_{(0)}$ is the generator of Brownian motion on $(0, \infty)$, and $G_{(\kappa)}$ is the generator of $\kappa + 1$ dimensional Bessel process for $\kappa \in \mathbb{N}$. We may set the scale function $s_{(\kappa)}$ and the speed measure $m_{(\kappa)}$ as follows.

$$ds_{(\kappa)}(x) = x^{-\kappa} \, dx, \quad dm_{(\kappa)}(x) = 2x^\kappa \, dx.$$  

The end point $0$ is $(s_{(\kappa)}, m_{(\kappa)}, 0)$-exit, or regular, or entrance according to $\kappa \leq -1$, or $-1 < \kappa < 1$, or $1 \leq \kappa$. The end point $\infty$ is always natural. The $\alpha$-Green function $G(\alpha, x, y)$ corresponding to $G(\kappa)$ is given by the following.

$$G(\kappa)(\alpha, x, y) = G(\kappa)(\alpha, y, x) = \begin{cases} (xy)^\mu I_{|\mu|}(\sqrt{2\alpha} x) K_{|\mu|}(\sqrt{2\alpha} y), & \text{if } \alpha > 0, \\ (2\mu)^{-1} x^{2\mu}, & \text{if } \kappa < 1, \alpha = 0, \end{cases}$$

for $0 < x \leq y < \infty$, where $I_{\nu}$ and $K_{\nu}$ are the modified Bessel functions defined by the following.

$$I_{\nu}(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad \nu > -1,$$

$$K_{\nu}(x) = K_{-\nu}(x) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin \nu\pi}, & \nu \in \mathbb{Z}, \\ \frac{(-1)^\nu}{2} \lim_{\xi \to \nu} \left( \frac{\partial I_{-\nu}(\xi)}{\partial \xi} - \frac{\partial I_{\nu}(\xi)}{\partial \xi} \right), & \nu \notin \mathbb{Z}. \end{cases}$$

The elementary solution $p_{(\kappa)}(t, x, y)$ is given as follows.

$$p_{(\kappa)}(t, x, y) = \frac{1}{2t} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} (xy)^\mu I_{|\mu|} \left( \frac{xy}{t} \right), \quad t, x, y > 0. \quad (6.9)$$

We note the following formula (see [6], p.200).

$$\int_0^\infty e^{-\lambda t} J_\nu(\sqrt{\lambda} x) J_\nu(\sqrt{\lambda} y) \, d\lambda = \frac{1}{t} \exp \left\{ -\frac{x^2 + y^2}{4t} \right\} I_\nu \left( \frac{xy}{2t} \right), \quad (6.10)$$

for $t, x, y > 0$ and $\nu > -1$, where $J_\nu$ is the Bessel function defined by

$$J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)}.$$
By means of (6.9) and (6.10), we get the following representation.

\[ p_{(\kappa)}(t, x, y) = \int_{0}^{\infty} e^{-\lambda t} \psi_{(\kappa)}(x, \lambda) \psi_{(\kappa)}(y, \lambda) \sigma_{(\kappa)}(\lambda) \, d\lambda, \]

where

\[ \psi_{(\kappa)}(x, \lambda) = C_{(\kappa)}(\lambda) x^{\mu} J_{[\mu]}(\sqrt{2} \lambda x), \quad \sigma_{(\kappa)}(\lambda) = C_{(\kappa)}(\lambda)^{-2}, \]

and \( C_{(\kappa)}(\lambda) \) is a positive number satisfying

\[ \lim_{x \to 0} \psi_{(\kappa)}(x, \lambda) = 1 \text{ if } \kappa \geq 1, \]
\[ \lim_{x \to 0} D_{s_{(\kappa)}} \psi_{(\kappa)}(x, \lambda) = 1 \text{ if } \kappa < 1, \]

so that,

\[ C_{(\kappa)}(\lambda) = \begin{cases} 2^{-1} (\lambda/2)^{-\mu/2} \Gamma(\mu), & \text{if } \kappa < 1, \\ (\lambda/2)^{-|\mu|/2} \Gamma(1 + |\mu|), & \text{if } \kappa \geq 1. \end{cases} \]

For \( \beta > 0 \), we set

\[ h_{(\kappa)}(x) = \left( \sqrt{2\beta} x \right)^{\mu} K_{|\mu|} \left( \sqrt{2\beta} x \right). \] (6.11)

The function \( h_{(\kappa)} \) is positive and satisfies \( G_{(\kappa)} h_{(\kappa)} = \beta h_{(\kappa)} \). We denote by \( \tilde{G}_{(\kappa)} \) the transform of \( G_{(\kappa)} \) with respect to \( h_{(\kappa)} \), that is,

\[ \tilde{G}_{(\kappa)} = \frac{1}{2} d^{2} + \left( \frac{\kappa}{2x} + \frac{h_{(\kappa)}'(x)}{h_{(\kappa)}(x)} \right) \frac{d}{dx} \]
\[ = \frac{1}{2} d^{2} + \left( \frac{1}{2x} + \sqrt{2\beta} \frac{K_{|\mu|}(\sqrt{2\beta} x)}{K_{[\mu]}(\sqrt{2\beta} x)} \right) \frac{d}{dx}. \] (6.12)

Since \( h_{(\kappa)}(0) \in (0, \infty) \) [resp. = \( \infty \)] if \( \kappa \in (-\infty, 1) \) [resp. \( \in [1, \infty] \)], and \( |\tilde{m}_{(\kappa)}(0)| \in [0, \infty) \) [resp. = \( \infty \)] if \( \kappa \in [1, 3) \) [resp. \( \in [3, \infty] \)], the end point 0 is \( (\tilde{s}_{(\kappa)}, \tilde{m}_{(\kappa)}, 0) \)-regular or exit according to \( \kappa < 3 \) or \( \kappa \geq 3 \), where

\[ d\tilde{s}_{(\kappa)}(x) = h_{(\kappa)}(x)^{-2} x^{-\kappa} \, dx, \quad d\tilde{m}_{(\kappa)}(x) = 2 h_{(\kappa)}(x)^{2} x^{\kappa} \, dx. \] (6.13)
We denote by \( \widetilde{p}(\kappa)(t, x, y) \) the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( \widetilde{G}(\kappa) \). Then we get the following.

\[
\begin{align*}
\widetilde{p}(\kappa)(t, x, y) &= e^{-\beta t} p(\kappa)(t, x, y)/h(\kappa(x)h(\kappa(y)), \\
&= \frac{1}{2(2\beta)^\mu t} \exp\left\{-\beta t - \frac{x^2 + y^2}{2t}\right\} I_{|\mu|}\left(\frac{xy}{t}\right)/K_{|\mu|}(\sqrt{2\beta x})K_{|\mu|}(\sqrt{2\beta y}) \\
&= \int_\beta^\infty e^{-\lambda} \widetilde{\psi}(\kappa)(x, \lambda)\widetilde{\psi}(\kappa)(x, \lambda)\tilde{\sigma}(\kappa)(\lambda) d\lambda,
\end{align*}
\]

(6.14)

where \( \widetilde{\psi}(\kappa)(x, \lambda) \) and \( \tilde{\sigma}(\kappa)(\lambda) \) are obtained by means of (5.13), (5.14), (5.17) and (5.18).

We now take a \( \kappa_1 \in (-\infty, 1) \) and put \( \kappa_2 = 2 - \kappa_1 \in (1, \infty) \). Then \( |(1 - \kappa_1)/2| = |(1 - \kappa_2)/2| \). We thus find that \( G(\kappa_1) \neq G(\kappa_2) \), but \( \widetilde{G}(\kappa_1) = \widetilde{G}(\kappa_2) \). This also shows that \( h \) transform of ODGDO is not one-to-one correspondence.

**Example 6.3** Finally we consider the following ODGDO \( G_o \) on \((0, \infty)\).

\[
G_o = \frac{1}{2} d^2 x^2 - cx^{-2},
\]

(6.15)

where \( c \) is a positive number. We may set

\[
ds_o(x) = dx, \quad dm_o(x) = 2dx, \quad dk_o(x) = 2cx^{-2} dx.
\]

We note that both of the end points 0 and \( \infty \) are \((s_o, m_o, k_o)\)-natural.

Let \( p_o(t, x, y) \) be the elementary solution of the equation (1.1) with \( G_{s,m,k} \) replaced by \( G_o \). By using Proposition 3.3, we show that \( p_o(t, x, y) \) is given by (6.19) below.

For \( \beta > 0 \), we set

\[
h_o(x) = \sqrt{x} K_\nu(\sqrt{2\beta x}),
\]

where \( \nu = \sqrt{2c+1/4} > 1/2 \). The function \( h_o \) is positive and satisfies \( G_o h_o = \beta h_o \). We denote by \( \tilde{G}_o \) the \( h \) transform of \( G \) with respect to \( h_o \). Then we find

\[
\tilde{G}_o = \frac{1}{2} d^2 x^2 + \frac{h_o'(x)}{h_o(x)} \frac{d}{dx} \\
= \frac{1}{2} d^2 x^2 + \left\{ \frac{1}{2} + \sqrt{2\beta} K_\nu'\left(\sqrt{2\beta x}\right) K_\nu\left(\sqrt{2\beta x}\right) \right\} \frac{d}{dx}.
\]

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This shows that $\tilde{G}_o$ coincides with $\tilde{G}_{(\kappa)}$ with $\nu = |\mu| = |(1 - \kappa)/2|$ in the preceding example. It should be noted that we take the scale $\tilde{s}_o$ [resp. $\tilde{s}_{(\kappa)}$] and the speed measure $\tilde{m}_o$ [resp. $\tilde{m}_{(\kappa)}$] corresponding to $\tilde{G}_o$ [resp. $\tilde{G}_{(\kappa)}$], where
\begin{equation}
\begin{aligned}
d\tilde{s}_o(x) &= h_o(x)^{-2} \, dx, \\
d\tilde{m}_o(x) &= 2h_o(x)^2 \, dx.
\end{aligned}
\end{equation}
Let $\tilde{p}_o(t, x, y)$ be the elementary solution of the equation (1.1) with $G_{s,m,k}$ replaced by $\tilde{G}_o$. Then we have
\begin{equation}
\tilde{p}_o(t, x, y) = e^{-\beta t} p_o(t, x, y) / h_o(x) h_o(y).
\end{equation}
Let $\nu = |(1 - \kappa)/2|$. Since $\tilde{G}_o$ coincides with $\tilde{G}_{(\kappa)}$, noting (6.16) and (6.13), we see that
\begin{equation}
\tilde{p}_o(t, x, y) h_o(y)^2 = \tilde{p}_{(\kappa)}(t, x, y) h_{(\kappa)}(y)^2 y^\nu.
\end{equation}
By using (6.9), (6.11), (6.14), (6.17), and (6.18), we obtain the following.
\begin{equation}
p_o(t, x, y) = p_{(\kappa)}(t, x, y) \frac{h_{(\kappa)}(y)}{h_{(\kappa)}(x)} \frac{h_o(x)}{h_o(y)} y^\nu = \frac{1}{2t} \exp \left\{ - \frac{x^2 + y^2}{2t} \right\} (xy)^{1/2} I_\nu \left( \frac{xy}{t} \right).
\end{equation}

References


