Elementary solutions of Bessel processes with boundary conditions

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1 Introduction

Let $L_\ast$ be the diffusion operator on an interval $I = (l_1, l_2)$, $0 < l_1 < l_2 \leq \infty$, defined by

$$L_\ast = \frac{1}{2} \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right),$$

which is the generator of a Bessel process. It is well known that, if $l_1 = 0$ [resp. $l_2 = \infty$], it is entrance [resp. natural] in the sense of Feller[1]. If $0 < l_i < \infty$ $(i = 1, 2)$, then $l_i$ is regular in the same sense as above. Therefore a boundary condition must be posed at $l_i$. In this paper we consider three kinds of boundary conditions, that is, absorbing, reflecting, and elastic boundary conditions.

Let $p_{(l_1, l_2)}^{(\alpha_1, \alpha_2)}(t, x, y)$ be the elementary solution of the equation

$$\frac{\partial u(t, x)}{\partial t} = L_\ast u(t, x), \quad t > 0, \ x \in I,$$

in the sense of McKean[5], where $\alpha_i \in \{A, R, L, E, N\}, i = 1, 2,$ and $\alpha_1$ and $\alpha_2$ denote state of boundaries $l_1$ and $l_2$, respectively, that is, $\alpha_i = A$ means that $l_i$ is regular and absorbing, $\alpha_i = R$ means that $l_i$ is regular and reflecting, $\alpha_i = L$ means that $l_i$ is regular and elastic, and further $\alpha_i = E$ means that $l_i$ is entrance (and hence $i = 1$ and $l_1 = 0$), $\alpha_i = N$ means that $l_i$ is natural (and hence $i = 2$ and $l_2 = \infty$).

The aim of this paper is to give explicit spectral representations of $p_{(l_1, l_2)}^{(\alpha_1, \alpha_2)}(t, x, y)$ for $\alpha_i \in \{A, R, L, E, N\}, i = 1, 2$. In the next section we state our results. In order to show them, we follow the same argument as in [4], [5], [6], [9], [10], which is summarized in Sect. 3. Sect. 4 is devoted to their proofs.

2 Main results

Let $I = (l, r), dm$ be a nonnegative Borel measure on $I$, which is finite on each compact set of $I$, and $s$ be a continuous increasing function on $I$. For an arbitrarily fixed point $c \in I$, we set $m(x) = m((c, x])$, $m(l) = m(+(c) \in [-\infty, 0]$, $m(l) = m(l-) \in [0, \infty]$, $s(l) = s(l) \in [-\infty, \infty]$ and $s(r) = s(r-) \in (-\infty, \infty]$. For a function $f$ on $I$, $D_s f$ stands for the right derivative of $f$ with respect to $s$ if it exists, that is, $D_s f(x) = \lim_{\epsilon \to 0}(f(x + \epsilon) - f(x))/\{s(x + \epsilon) - s(x)\}$. We set

$$I^* = I \cup \{x; x = l \text{ with } |s(l)| + |m(l)| < \infty \text{ or } x = r \text{ with } |s(r)| + |m(r)| < \infty\},$$

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\[ I_*(m) = \{ x \in I; m(x_1) < m(x_2) \text{ for } l < x_1 < x < x_2 < r\}, \]
\[ I_*(m) = I_*(m) \cup \{ x; x = l \text{ with } |s(l)| + |m(l)| < \infty \]
\[ \text{or } x = r \text{ with } |s(r)| + |m(r)| < \infty \} \]

Throughout this paper we assume that
\[ I(m) \neq \emptyset, \quad \inf I_*(m) = l, \quad \sup I_*(m) = r. \]

Let \( C_b(E) \) be the set of all bounded continuous function on \( E \), where \( E \) is a Borel set. Let \( D(L) \) be the space of all functions \( u \in C_b(I^*) \) for which there exists a function \( f \in C_b(I_*(m)) \) satisfying the following conditions.

\( (C.1) \) There are two constants \( A_1, A_2 \) such that
\[ u(x) = A_1 + A_2 \{ s(x) - s(c) \} + \int_{(c, x]} \{ s(x) - s(y) \} f(y) \, dm(y), \quad x \in I. \]

\( (C.2) \) If \( |s(l)| + |m(l)| < \infty \), then
\[ \theta_1 u(l) - \theta_2 D_s u(l+) = 0, \]
where \( \theta_1, \theta_2 \) are nonnegative numbers satisfying \( \theta_1 + \theta_2 = 1 \).

\( (C.3) \) If \( |s(r)| + |m(r)| < \infty \), then
\[ \theta_1 u(r) + \theta_2 D_s u(r-) = 0, \]
where \( \theta_1, \theta_2 \) are nonnegative numbers satisfying \( \theta_1 + \theta_2 = 1 \).

\( (3) \) with \( \theta_2 = 0 \) [resp. (4) with \( \theta_2 = 0 \)] is called the absorbing boundary condition at \( l \) [resp. \( r \)]. \( (3) \) with \( \theta_1 > 0 \) and \( \theta_2 > 0 \) [resp. (4) with \( \theta_1 > 0 \) and \( \theta_2 > 0 \)] is called the elastic boundary condition at \( l \) [resp. \( r \)]. \( (3) \) with \( \theta_1 = 0 \) [resp. (4) with \( \theta_1 = 0 \)] is called the reflecting boundary condition at \( l \) [resp. \( r \)]. \( |s(l)| + |m(l)| < \infty \) [resp. \( |s(r)| + |m(r)| < \infty \)] implies that \( l \) [resp. \( r \)] is regular in the sense of Feller [1]. In the next section we precisely state the classification of boundaries due to Feller. The operator \( L \) is defined by the mapping from \( u \in D(L) \) to \( f \in C_b(I_*(m)) \). \( m \) and \( s \) are called the speed measure and the scale function for \( C \), respectively.

Let \( \mathbb{D} = \{ X(t); t \geq 0, P_x; x \in I_* \} \) be a generalized diffusion process whose generator is given by \( L \). Then it is known that there exists the transition probability density \( p(t, x, y) \) such that
\[ P_x(X(t) \in E) = \int_E p(t, x, y) \, dm(y), \]
for \( t > 0, x \in I_*(m), E \in B(I_*(m)), \) where \( B(E) \) stands for the set of all Borel sets of \( E \) ([3]). The function \( p(t, x, y) \) is the elementary solution of the equation (2) with \( L \) in place of \( L_* \). Note that \( p(t, x, y) = p(t, y, x) \) and \( p(t, x; y) \) is positive and continuous for \( t > 0, x, y \in I \).
Now we state our result. We go back to the diffusion operator \( L_* \) on \((l_1, l_2)\) defined by (1). It is easy to see that the scale function \( s_* \) and the speed measure \( m_* \) for \( L_* \) are given by

\[
s_*(x) = C^{-1} \log x, \quad dm_*(x) = 2C x \, dx,
\]

where \( C \) is a positive constant. We may set \( C = 1 \) without loss of generality. Our results are represented by means of Bessel functions \( J_\nu(x) \), \( N_\nu(x) \) and modified Bessel functions \( I_\nu \), \( K_\nu \) with \( \nu = 0, 1 \), which are given as follows. For \( x > 0 \),

\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)},
\]

\[
N_\nu(x) = 2\pi J_\nu(x) \left( \gamma + \log \frac{x}{2} \right)
\]

\[
- \frac{1}{\pi} \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)} \left[ \sum_{m=1}^{n \nu + \nu + 1} \frac{1}{m} \right]
\]

\[
I_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)},
\]

\[
K_\nu(x) = (-1)^\nu + 1 I_\nu(x) \left( \gamma + \log \frac{x}{2} \right)
\]

\[
+ \frac{(-1)^\nu}{2} \sum_{n=0}^{\infty} \frac{(-x/2)^{2n}}{n! (\nu + n)!} \left[ \sum_{m=1}^{n \nu + \nu + 1} \frac{1}{m} \right]
\]

\[
+ \frac{1}{2} \sum_{r=1}^{\nu} (-1)^r \frac{(-x/2)^{2n}}{r!} \left( \frac{x}{2} \right)^{2r} \nu,
\]

where \( \gamma \) is Euler's constant and \( \gamma = 0.57721 \ldots \).

It is well known that the elementary solution \( p_{(0, \infty)}^{EN}(t, x, y) \) is given by

\[
p_{(0, \infty)}^{EN}(t, x, y) = \frac{1}{2\sqrt{\pi t}} e^{-(x^2+y^2)/2t} I_0(xy/t)
\]

\[
= \frac{1}{2} \int_0^\infty e^{-\lambda} J_0(\sqrt{2\lambda x}) J_0(\sqrt{2\lambda y}) \, d\lambda.
\]

First we consider the case that \( l_1 = 0 \) and \( l_2 = a \in (0, \infty) \).

**PROPOSITION 1.** Let \( l_1 = 0 \) and \( l_2 = a \in (0, \infty) \). Assume the boundary condition (4) with \( r, \theta_1^r \) and \( \theta_2^r \) replaced by \( a, \theta_1 \) and \( \theta_2 \), respectively. Then \( p^{EA}(t, x, y) \) is given by

\[
p_{(0, a)}^{EA}(t, x, y) = \frac{n^2}{4} \sum_{\zeta \in \Gamma(\theta_1, \theta_2)} \zeta^2 \left( \theta_1 N_0(\zeta a) - \zeta a \theta_2 N_1(\zeta a) \right)^2 \frac{\theta_1^2 + \zeta^2 a^2 \theta_2^2}{\theta_1^2 + \zeta^2 a^2 \theta_2^2} e^{-\zeta^2 t/2} J_0(\zeta x) J_0(\zeta y),
\]
where $\alpha = A$ or $L$ or $R$; $\theta_2 = 0$ if $\alpha = A; \theta_1 = 0$ if $\alpha = R$; $K(\theta_1, \theta_2) = \{\zeta \in [0, \infty); \theta_1 J_0(\zeta a) + \theta_2 J_1(\zeta a)\}$. 

We note the formula (6) with $a = 1$ and $\theta_2 = 0$ is given in Sect. 5 of [2].

We next consider the case that $l_1 = a \in (0, \infty)$ and $l_2 = \infty$. In this case the spectrum only consists of continuous one. Let

\begin{align}
F(\lambda; \xi; \eta) &= N_0(\sqrt{2\lambda \xi}) J_1(\sqrt{2\lambda \eta}) - J_0(\sqrt{2\lambda \xi}) N_1(\sqrt{2\lambda \eta}), \quad (7) \\
G_\nu(\lambda; \xi; \eta) &= J_\nu(\sqrt{2\lambda \xi}) N_\nu(\sqrt{2\lambda \eta}) - N_\nu(\sqrt{2\lambda \xi}) J_\nu(\sqrt{2\lambda \eta}), \quad (8)
\end{align}

for $\lambda, \xi, \eta \in (0, \infty)$ and $\nu = 0, 1$.

**PROPOSITION 2.** Let $l_1 = a \in (0, \infty)$ and $l_2 = \infty$. Assume the boundary condition (3) with $a$, $\theta_1$, and $\theta_2$ in place of $1$, $\theta_1'$, and $\theta_2'$, respectively. Then $p_{(a, \infty)}^{N}(t, x, y)$ is given by

\begin{equation}
p_{(a, \infty)}^{N}(t, x, y) = \int_0^\infty e^{-\lambda t} \phi_{(a, \infty)}^{N}(x, \lambda) \phi_{(a, \infty)}^{N}(y, \lambda) \sigma_{(a, \infty)}(\lambda) \, d\lambda,
\end{equation}

where $\alpha = A$ or $L$ or $R$; $\theta_2 = 0$ if $\alpha = A; \theta_1 = 0$ if $\alpha = R$.

\begin{align}
\phi_{(a, \infty)}^{N}(x, \lambda) &= \frac{\pi}{2} \left\{ \theta_1 G_0(\lambda; a; x) + \theta_2 \sqrt{2\lambda a} F(\lambda; x; a) \right\}, \\
\sigma_{(a, \infty)}(\lambda) &= \left[ \left\{ \theta_1 N_0(\sqrt{2\lambda a}) + \theta_2 \sqrt{2\lambda a} N_1(\sqrt{2\lambda a}) \right\}^2 \\
&\quad + \left\{ \theta_1 J_0(\sqrt{2\lambda a}) + \theta_2 \sqrt{2\lambda a} J_1(\sqrt{2\lambda a}) \right\}^2 \right]^{-1}.
\end{align}

We finally consider the case that $l_1 = a \in (0, \infty)$ and $l_2 = b \in (0, \infty)$.

**PROPOSITION 3.** Let $l_1 = a \in (0, \infty)$ and $l_2 = b \in (0, \infty)$. Assume that the boundary condition (3) with $a$, $\theta_1$, and $\theta_2$ in place of $1$, $\theta_1'$, and $\theta_2'$, and the boundary condition (4) with $b$, $\theta_1^b$ and $\theta_2^b$ in place of $r$, $\theta_1^r$ and $\theta_2^r$. Then $p_{(a, b)}^{\alpha}(t, x, y)$ is given by

\begin{equation}
p_{(a, b)}^{\alpha}(t, x, y) = \sum_{\zeta \in \{\theta_1^a, \theta_2^a, \theta_1^b, \theta_2^b\}} e^{-\zeta t} \phi_{(a, b)}^{\alpha}(x, \zeta) \phi_{(a, b)}^{\alpha}(y, \zeta) \sigma_{(a, b)}^\alpha(\zeta),
\end{equation}

where

\begin{align}
\phi_{(a, b)}^{\alpha}(x, \zeta) &= \frac{\theta_2^\alpha}{\theta_1^\alpha} \sqrt{2\zeta a \pi} F(\zeta; x; a) + \frac{\pi}{2} G_0(\zeta; a; x), \quad \text{if } \alpha = A \text{ or } L, \\
\phi_{(a, b)}^{R\beta}(x, \zeta) &= \sqrt{2\zeta a \pi} F(\zeta; x; a), \\
\sigma_{(a, b)}^\alpha(\zeta) &= \left[ \frac{1}{2\zeta} - \left( \frac{\theta_2^a}{\theta_1^a} \right)^2 \right]^{-1} \\
&\quad + \frac{\left\{ \theta_1^b \theta_2^\alpha \right\}^2 + 2 \left( \theta_2^b \right)^2 \theta_1^b \theta_2^\alpha F(\zeta; a; b) + \theta_2^b \sqrt{2\zeta a} G_1(\zeta; a; b)}{2\lambda a \theta_1^a \theta_2^\alpha} \theta_1^b F(\zeta; b; a) - \theta_2^b \sqrt{2\zeta b} G_1(\zeta; a; b)}^{-1} \quad \text{if } \alpha = A \text{ or } L.
\[
\sigma^{R\beta}_{(a,b)}(\zeta) = \left[ -a^2 + ab \theta^2_1 \pi G_1(\zeta; a; b) + \theta^2_2 \sqrt{2\zeta} b F(\zeta; a; b) \right]^{-1},
\]
\[
L(\theta_1^0, \theta_2^0, \theta_1^0, \theta_2^0) = \left\{ \zeta \in [0, \infty); \theta_2^0 \left[ \theta_1^0 \sqrt{2\zeta} a F(\zeta; b; a) - 2\theta_2^0 ab \zeta G_1(\zeta; a; b) \right] \right\}.
\]

The following result is a direct consequence of Propositions 1, 2 and 3.

**COROLLARY 4.** The following formulae hold true.

\[
\lim_{a \to 0} p^{AN}_{(a, \infty)}(t, x, y) = p^{EN}_{(0, \infty)}(t, x, y).
\]
\[
\lim_{a \to 0} p^{\alpha\beta}_{(a, b)}(t, x, y) = p^{E\beta}_{(0, b)}(t, x, y),
\]

where \( \alpha, \beta \in \{A, L, R\} \).

This result is also obtained by using a convergence theorem on a sequence of elementary solutions due to Ogura [7].

### 3 Preliminaries

In this section we summarize some facts on elementary solutions, which are discussed in [3], [4], [5], [6], [9], [10], etc. Let \( \mathcal{L} \) be the operator defined at the beginning of the preceding section. Let \( m \) and \( s \) be the speed measure and the scale function for \( \mathcal{L} \), respectively. We introduce the following two quantities.

\[
J_t^{l, \nu} = \int_{(l, r]} dv(x) \int_{[x, c]} d\mu(y),
\]
\[
J^r_{\mu, \nu} = \int_{[l, r]} dv(x) \int_{[x, c]} d\mu(y),
\]

where \( d\mu \) and \( dv \) are Borel measure on \( I = (l, r) \) and \( c \in I_*(m) \). We note that \( J_t^{l, \nu} \) [resp. \( J^r_{\mu, \nu} \)] is finite for some \( c \in I_*(m) \) if and only if \( J_t^{l, \nu} \) [resp. \( J^r_{\mu, \nu} \)] is finite for each \( c \in I_*(m) \). For \( a = l, r \), \( a \) is called to be

- **regular** if \( J_m^a < \infty, J_s^a < \infty \),
- **exist** if \( J_m^a = \infty, J_s^a < \infty \),
- **entrance** if \( J_m^a < \infty, J_s^a = \infty \),
- **natural** if \( J_m^a = \infty, J_s^a = \infty \).

It is easy to see that, for each \( a = l, r \),

\[
|m(a)| + |s(a)| < \infty \quad \text{if } a \text{ is regular},
\]
\[
|m(a)| = \infty, |s(a)| < \infty \quad \text{if } a \text{ is exist},
\]
\[
|m(a)| < \infty, |s(a)| = \infty \quad \text{if } a \text{ is entrance},
\]
\[
|m(a)| = \infty, |s(a)| = \infty \quad \text{if } a \text{ is natural}.
\]
In the rest of this section we assume that \( l \) is regular. We define the elementary solution \( p(t, x, y) \) of the generalized diffusion equation (2) with \( L \) replaced by \( L \).

Here and hereafter we use the conventions \( 1/\infty = 0 \) and \( \pm a/0 = \pm \infty \) for a positive constant \( a \). We put

\[
\tilde{r} = \begin{cases} 
  r & \text{if } |m(r)| + |s(r)| = \infty, \\
  r + \theta_2/\theta_1 & \text{if } |m(r)| + |s(r)| < \infty,
\end{cases}
\]

\[
m(x) = \begin{cases} 
  m(l), & l \leq x < \tilde{l}, \\
  m(x), & \tilde{l} \leq x < r, \\
  m(r), & r \leq x < \tilde{r}, \\
  \infty, & \tilde{r} \leq x,
\end{cases}
\]

\[S_m = (\tilde{l}, \tilde{r}).\]

3.1 The case that \( \tilde{l} > -\infty \).

We note that \( \tilde{l} > -\infty \) implies that \( l \) is absorbing or elastic with \( \theta_l^l > 0 \). Let \( \varphi_i(x, \alpha), i = 1, 2, \alpha \in \mathbb{C}, \) be the solutions of the integral equations

\[
\varphi_1(x, \alpha) = 1 + \alpha \int_{(1, x]} \{s(x) - s(y)\} \varphi_1(y, \alpha) \, dm(y), \quad x \in S_m,
\]

\[
\varphi_2(x, \alpha) = s(x) - s(\tilde{l}) + \alpha \{s(x) - s(y)\} \varphi_2(y, \alpha) \, dm(y), \quad x \in S_m.
\]

Then for each \( \alpha > 0 \), there exists the limit

\[
k(\alpha) = \lim_{\tilde{x} \to x} (\alpha) = \varphi_2(x, \alpha)/\varphi_1(x, \alpha).
\]

The function \( k(\alpha) \) can be analytically continued to \( \mathbb{C} \setminus (-\infty, 0] \). The spectral measure \( \sigma_0 \) is defined by

\[
\sigma_0([\lambda_1, \lambda_2]) = -\lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{1}{k(-\lambda - \sqrt{-1} \epsilon)} \, d\lambda,
\]

for all continuity points \( \lambda_1 \) and \( \lambda_2 \) of \( \sigma_0 (\lambda_1 < \lambda_2) \). We define the elementary solution of the generalized diffusion equation (2) by

\[
p(t, x, y) = \int_{(0, \infty)} e^{-\lambda t} \varphi_2(x, -\lambda) \varphi_2(y, -\lambda) \sigma_0(d\lambda),
\]

for \( t > 0, \ x, y \in S_m. \)
3.2 The case that \( \dot{t} = -\infty \).

This is the case that \( l \) is reflecting. Let \( \psi_i(x, \alpha), i = 1, 2, \alpha \in \mathbb{C} \), be the solutions of the integral equations

\[
\psi_1(x, \alpha) = 1 + \alpha \int_{(l, x]} \{ s(x) - s(y) \} \psi_1(y, \alpha) \, dm(y), \quad x \in S_m, \tag{15}
\]

\[
\psi_2(x, \alpha) = s(x) - s(l) + \alpha \int_{(l, x]} \{ s(x) - s(y) \} \psi_2(y, \lambda) \, dm(y), \quad x \in S_m. \tag{16}
\]

Then for each \( \alpha > 0 \), there exists the limit

\[
h(\alpha) = \lim_{x \uparrow x} \psi_2(x, \alpha) / \psi_1(x, \alpha). \tag{17}
\]

The function \( h(\alpha) \) can be analytically continued to \( \mathbb{C} \setminus (-\infty, 0] \). The spectral measure \( \sigma^0 \) is defined by

\[
\sigma^0(\lambda_1, \lambda_2) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im h(-\lambda - \sqrt{-1} \epsilon) \, d\lambda,
\]

for all continuity points \( \lambda_1 \) and \( \lambda_2 \) of \( \sigma^0(\lambda_1 < \lambda_2) \). We define the elementary solution of the generalized diffusion equation (2) by

\[
p(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi_1(x, -\lambda) \psi_1(y, -\lambda) \sigma^0(d\lambda), \tag{18}
\]

for \( t > 0, x, y \in S_m \).

4 Spectral representations of elementary solutions

In this section we prove Propositions 1, 2 and 3. Throughout this section, we set

\[
\Phi_a(x, \alpha) = \sqrt{2\alpha a} \{ K_1(\sqrt{2\alpha a}) I_0(\sqrt{2\alpha x}) + I_1(\sqrt{2\alpha a}) K_0(\sqrt{2\alpha x}) \}, \tag{19}
\]

\[
\Phi_a(x, -\alpha) = \frac{\sqrt{2\alpha a \pi}}{2} F(\alpha; x; a), \tag{20}
\]

\[
\Psi_a(x, \alpha) = K_0(\sqrt{2\alpha a}) I_0(\sqrt{2\alpha x}) - I_0(\sqrt{2\alpha a}) K_0(\sqrt{2\alpha x}), \tag{21}
\]

\[
\Psi_a(x, -\alpha) = -\pi G_0(\alpha; x) \tag{22}
\]

for \( x > 0 \) and \( \alpha > 0 \), where \( a \) is a positive number. It is easy to see that

\[
\Phi_a(a, \alpha) = 1, \quad \frac{\partial}{\partial x} \Phi_a(x, \alpha) \big|_{x=a} = 0,
\]

\[
\Psi_a(a, \alpha) = 0, \quad \frac{\partial}{\partial x} \Psi_a(x, \alpha) \big|_{x=a} = \frac{1}{a}.
\]
4.1 The Case that \( l_1 = 0 \) and \( l_2 = a \in (0, \infty) \).

We prove Proposition 1. Then \( a \) is regular, and satisfies the boundary condition (4) with \( r = a, \theta_1^r = \theta_1 \) and \( \theta_2^r = \theta_2 \). We set

\[
m(x) = \begin{cases} 
    -\infty, & x \leq 0, \\
    x^2, & 0 < x < a, \\
    a^2, & a \leq x < \bar{r}, \\
    \infty, & \bar{r} \leq x,
\end{cases}
\]

\[
s(x) = \begin{cases} 
    \log x, & 0 < x < a, \\
    x - a + \log a, & a \leq x \leq \bar{r}.
\end{cases}
\]

We apply the argument in the preceding section exchanging the role of \( l_1 \) and \( l_2 \).

Assume that \( a \) is absorbing or elastic. Then \( \varphi_1(x, \alpha), \varphi_2(x, \alpha), \) and \( k(\alpha) \) corresponding to (11), (12) and (13) are given as follows.

\[
\varphi_1(x, \alpha) = \begin{cases} 
    \Phi_a(x, \alpha), & 0 < x \leq a, \\
    1, & a < x < \bar{r},
\end{cases}
\]

\[
\varphi_2(x, \alpha) = \begin{cases} 
    -\Psi_a(x, \alpha) + (\theta_2/\theta_1)\Phi_a(x, \alpha), & 0 < x \leq a, \\
    \bar{r} - x, & a < x \leq \bar{r},
\end{cases}
\]

\[
k(\alpha) = \lim_{x \to 0} \frac{\varphi_2(x, \alpha)}{\varphi_1(x, \alpha)} = \frac{\theta_2}{\theta_1} + \frac{1}{\sqrt{2\alpha a}} \frac{I_0(\sqrt{2\alpha a})}{I_1(\sqrt{2\alpha a})}, \quad \alpha > 0.
\]

Let us fix \( \zeta \in \Lambda(\theta_1, \theta_2) := \{ \zeta \in (0, \infty) ; \theta_1 J_0(\sqrt{2\zeta a}) = \theta_2 J_1(\sqrt{2\zeta a}) \} \). Then, by means of Proposition A, there exists a positive \( \delta > 0 \) such that

\[
\sigma_0([\lambda_1, \lambda_2]) = -\lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \left( \frac{1}{k(-\lambda - \sqrt{-1}\epsilon)} \right) d\lambda = \frac{2\zeta}{1 + 2(\theta_2/\theta_1)^2 a^2 \zeta},
\]

for \( \zeta - \delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta \). By means of (20) and (22), (14) is reduced to

\[
\sum_{\zeta \in \Lambda(\theta_1, \theta_2)} e^{-\zeta \lambda} \varphi_2(x, -\zeta) \varphi_a(y, -\zeta) 2\zeta / \{1 + 2(\theta_2/\theta_1)^2 a^2 \zeta\} = \frac{\pi^2}{4} \sum_{\zeta \in \Delta(0)} \zeta^2 e^{-\zeta^2/2} N_0(\zeta a)^2 J_0(\zeta x) J_0(\zeta y),
\]

which shows (6) for \( \alpha = A \) or \( L \).

Next assume that \( a \) is reflecting. the \( \psi_1, \psi_2, \) and \( h \) corresponding to (15), (16) and (17) are given as follows.

\[
\psi_1(x, \alpha) = \Phi_a(x, \alpha), \quad \psi_2(x, \alpha) = -\Psi_a(x, \alpha), \quad 0 < x \leq a,
\]

\[
h(\alpha) = \lim_{x \to 0} \frac{\varphi_2(x, \alpha)}{\varphi_1(x, \alpha)} = \frac{1}{\sqrt{2\alpha a}} \frac{I_0(\sqrt{2\alpha a})}{I_1(\sqrt{2\alpha a})}.
\]

Let us fix \( \zeta \in [0, \infty) \) such that \( \sqrt{2\zeta} J_1(\sqrt{2\zeta a}) = 0 \). By means of Proposition A, there exists a positive \( \delta > 0 \) and

\[
\sigma^0([\lambda_1, \lambda_2]) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im h(-\lambda - \sqrt{-1}\epsilon) d\lambda = \frac{1}{a^2}.
\]
for $\zeta - \delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta$. By means of (20) and (22), (18) is reduced to
\[
\sum_{\zeta; \sqrt{\zeta} J_1((\sqrt{2}a)\zeta) = 0} \frac{e^{-t\psi_1(x, -\zeta)\psi_1(y, -\zeta)}}{a^2} = \frac{\pi^2}{4} \sum_{\zeta; J_1(\zeta) = 0} \zeta^2 e^{-\frac{\zeta^2}{2} N_1(\zeta a^2) J_0(\zeta x) J_0(\zeta y)},
\]
which shows (6) for $\alpha = R$.

### 4.2 The Case that $l_1 = a \in (0, \infty)$ and $l_2 = \infty$.

We prove Proposition 2. Then $a$ is regular and the boundary condition for (3) with $l = a$, $\theta_1 = \theta_2$ and $\theta_2 = \theta_1$. We set
\[
l = a - \frac{\theta_2}{\theta_1}, \quad \tilde{r} = \infty,
\]
\[
m(x) = \begin{cases} -\infty, & x \leq \tilde{l}, \\ \frac{a^2}{2}, & \tilde{l} < x < a, \\ x^2, & a \leq x, \end{cases} \quad s(x) = \begin{cases} x - a + \log a, & \tilde{l} < x \leq a, \\ \log x, & a < x. \end{cases}
\]

Assume that $a$ is absorbing or elastic. Then $\varphi_1, \varphi_2$ and $k$ define by (11), (12) and (13) are given as follows.
\[
\varphi_1(x, \alpha) = \Phi_\alpha(x, \alpha), \quad \varphi_2(x, \alpha) = \frac{\theta_2}{\theta_1} \Phi_\alpha(x, \alpha) + \Psi_\alpha(x, \alpha), \quad a \leq x < \infty,
\]
\[
k(\alpha) = \lim_{x \uparrow \tilde{r}} \frac{\varphi_2(x, \alpha)}{\varphi_1(x, \alpha)} = \frac{\theta_2}{\theta_1} + \frac{K_0(\sqrt{2}a\alpha)}{\sqrt{2}a K_1(\sqrt{2}a \alpha)}.
\]
Therefore
\[
\sigma_0([\lambda_1, \lambda_2]) = -\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \left( \frac{1}{k(-\lambda - \sqrt{-1}\epsilon)} - 1 \right) d\lambda
\]
\[
= \frac{2\theta_1}{\pi^2} \int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\{\theta_1 N_0(\sqrt{2}a\lambda) + \sqrt{2}a K_0(\sqrt{2}a \lambda)\}^2},
\]
for $0 < \lambda_1 < \lambda_2 < \infty$. Noting (20) and (22) we obtain (9) for $\alpha = A$ or $L$.

We next assume that $a$ is reflecting. The $\psi_1, \psi_2$, and $h$ defined by (15), (16) and (17) are given as follows.
\[
\psi_1(x, \alpha) = \Phi_\alpha(x, \alpha), \quad \psi_2(x, \alpha) = \Psi_\alpha(x, \alpha), \quad a \leq x < \infty,
\]
\[
h(\alpha) = \frac{K_0(\sqrt{2}a\alpha)}{\sqrt{2}a K_1(\sqrt{2}a \alpha)}.
\]
Therefore
\[
\sigma_0([\lambda_1, \lambda_2]) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im h(-\lambda - \sqrt{-1}\epsilon) d\lambda
\]
\[
= \int_{\lambda_1}^{\lambda_2} \frac{1}{2a^2 \lambda^2} \frac{1}{J_1(\sqrt{2}a \lambda)^2 + N_1(\sqrt{2}a \lambda)^2} d\lambda.
\]

Combining this with (22), we obtain (9) for $\alpha = R$.
4.3 The Case that \( l_1 = a \in (0, \infty) \) and \( l_2 = b \in (0, \infty) \).

We prove Proposition 3. Since \( a \) and \( b \) are regular, the boundary condition (3) is posed at \( a \), and (4) with \( r = b \) is posed at \( b \). We set

\[
\begin{align*}
\tilde{l} &= a - \frac{\theta_2^a}{\theta_1^a}, \\
\tilde{r} &= b + \frac{\theta_2^b}{\theta_1^b}, \\
m(x) &= \begin{cases} 
-\infty, & x \leq \tilde{l}, \\
a^2, & \tilde{l} \leq x < a, \\
x^2, & a \leq x < b, \\
b^2, & b \leq x \leq \tilde{r},
\end{cases}
\end{align*}
\]

where \( \tilde{x}(x) = \begin{cases} 
x - a + \log a, & \tilde{l} < x < a, \\
\log x, & a \leq x \leq b, \\
x - b + \log b, & b < x \leq \tilde{r}.
\end{cases} \)

We assume that \( a \) is absorbing or elastic. Then \( \varphi_1(x, \alpha), \varphi_2(x, \alpha) \) and \( k(\alpha) \) defined by (11), (12) and (13) are given as follows.

\[
\begin{align*}
\varphi_1(x, \alpha) &= \begin{cases} 
1, & \tilde{l} \leq x < a, \\
\Phi_a(x, \alpha), & a \leq x \leq b, \\
\Phi_a(b, \alpha) + (x - b)D_a \Phi_a(b, \alpha), & b < x < \tilde{r},
\end{cases} \\
\varphi_2(x, \alpha) &= \begin{cases} 
x - \tilde{l}, & \tilde{l} \leq x < a, \\
Q_a(x, \alpha) := \left( \frac{\theta_2^a}{\theta_1^a} \right) \Phi_a(x, \alpha) + \Psi_a(x, \alpha), & a \leq x < b, \\
Q_a(b, \alpha) + (x - b)D_a Q_a(b, \alpha), & b < x < \tilde{r},
\end{cases} \\
k(\alpha) &= \frac{\theta_2^a \Psi_a(b, \alpha) + \theta_2^b D_a \Psi_a(b, \alpha)}{\theta_1^a \Phi_a(b, \alpha) + \theta_2^b D_a \Phi_a(b, \alpha)}.
\end{align*}
\]

Let us fix \( \zeta \in \Lambda(\theta_1^a, \theta_2^a, \theta_1^b, \theta_2^b) := \{ \zeta \in (0, \infty) ; \theta_1^a \theta_1^b \sqrt{2\zeta} a F(\zeta; b; a) - 2\theta_1^b a b G_1(\zeta; a; b) = -\theta_1^b \theta_1^b G_0(\zeta; a; b) + \theta_1^b \sqrt{2\zeta} b F(\zeta; a; b) \} \). Then, by means of Proposition A, there exists a positive \( \delta > 0 \) such that

\[
\sigma_0([\lambda_1, \lambda_2]) = \left[ -\frac{1}{2\zeta} - \left( \frac{\theta_2^a a}{\theta_1^a} \right)^2 + \frac{\{(\theta_1^a)^2 + 2\zeta (\theta_1^a)^2\} b \theta_1^a F(\zeta; a; b) - \theta_2^b \sqrt{2\zeta} a G_1(\zeta; a; b)}{2\lambda a \theta_1^2 \theta_1^2 - \theta_1^2 F(\zeta; b; a) - \theta_2^2 \sqrt{2\zeta} b G_1(\zeta; a; b)} \right]^{-1}
\]

if \( \alpha = A \) or \( L \), for \( \zeta - \delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta \). By means of (20) and (22), we have (10) with \( \alpha = A \) or \( L \).

We next assume that \( a \) is reflecting. The \( \psi_1, \psi_2, \) and \( h \) defined by (15), (16) and (17) are given as follows.

\[
\begin{align*}
\psi_1(x, \alpha) &= \begin{cases} 
\Phi_a(x, \alpha), & \tilde{l} \leq x \leq b, \\
\Phi_a(b, \alpha) + (x - b)D_a \Phi_a(b, \alpha), & b < x \leq \tilde{r},
\end{cases} \\
\psi_2(x, \alpha) &= \begin{cases} 
x - \tilde{l}, & \tilde{l} \leq x < a, \\
\Psi_a(x, \alpha), & a \leq x < b, \\
\Psi_a(b, \alpha) + (x - b)D_a \Psi_a(b, \alpha), & b \leq x \leq \tilde{r},
\end{cases} \\
h(\alpha) &= \frac{\theta_1^a \Psi_a(b, \alpha) + \theta_2^b D_a \Psi_a(b, \alpha)}{\theta_1^a \Phi_a(b, \alpha) + \theta_2^b D_a \Phi_a(b, \alpha)}.
\end{align*}
\]
Let us fix $\zeta \in \Lambda(\theta_1^0, \theta_2^0) := \{ \zeta \in (0, \infty); \theta_1^0 \sqrt{2\kappa} a_F(\zeta; b; a) = 2b\kappa ab \zeta G_1(\zeta; a; b) \}$. Then, by means of Proposition A, there exists a positive $\delta > 0$ such that

$$\sigma_0([\lambda_1, \lambda_2]) = \left[ -a^2 + ab \frac{\theta_1^0 \pi G_1(\zeta, a, b) + \theta_2^0 \sqrt{2\kappa} b \pi F(\zeta, b, a)}{\theta_1^0 \pi G_0(\zeta, a, b) + \theta_2^0 \sqrt{2\kappa} ab F(\zeta, a, b)} \right]^{-1},$$

for $-\delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta$. By means of (20) and (22), we have (10) with $\alpha = R$.

5 Appendix

The following result is useful for calculations of spectrum. The proof is due to Dr. Sechiko Takahashi. We would like to thank her for suggesting improvements of our results.

**Proposition A** Let $k(z) : \mathbb{C} \rightarrow \mathbb{C}$ and $k(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Assume that there is a $\lambda_0 \in \mathbb{R}$ such that $k$ is analytic in a neighborhood of $\lambda_0$, $k(\lambda_0) = 0$, and it is a zero of order 1. Then there exists a positive $\delta$ such that

$$\lim_{\gamma \downarrow 0} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda)} d\lambda = \frac{\pi}{k'(\lambda_0)}.$$

**Proof.** By virtue of assumption of the proposition, there is a neighborhood $U$ of $\lambda_0$ such that $1/k$ is a simple pole of $1/k$, and $\varphi(z) := 1/k(z) - 1/(z - \lambda_0)k'(\lambda_0)$ is analytic in $U$. Let us fix a $\delta > 0$ such that $[\lambda_0 - \delta, \lambda_0 + \delta] \subset U$. Further fix an $\epsilon_0 > 0$ such that

$$L_1 := \{ \lambda_0 - \delta + \sqrt{-1}\epsilon; -\epsilon_0 < \epsilon < \epsilon_0 \} \subset U,$$
$$L_2 := \{ \lambda_0 + \delta + \sqrt{-1}\epsilon; -\epsilon_0 < \epsilon < \epsilon_0 \} \subset U.$$

Since $1/k$ is analytic on $L_1$ and $L_2$, there is an $M > 0$ such that $|k(z)| \leq M$ for $z \in L_1 \cup L_2$.

For $0 < \epsilon < \epsilon_0$ and $0 < \rho < \min\{\delta, \epsilon\}$, we set

$$\gamma_1 : z(\lambda) = \lambda + \sqrt{-1}\epsilon (\lambda_0 + \delta \geq \lambda \geq \lambda_0 - \delta), \quad \gamma_2 : z(t) = \lambda - \delta + \sqrt{-1}t (\epsilon \geq t \geq -\epsilon),$$
$$\gamma_3 : z(\lambda) = \lambda - \sqrt{-1}\epsilon (\lambda_0 - \delta \leq \lambda \leq \lambda_0 + \delta), \quad \gamma_4 : z(t) = \lambda - \delta + \sqrt{-1}t (-\epsilon \leq t \leq \epsilon),$$
$$\gamma : z = \rho e^{\sqrt{-1}\theta} \quad 0 \leq \theta \leq 2\pi \quad (\rho \leq \delta, \epsilon).$$

By virtue of Cauchy's integral theorem,

$$\sum_{i=1}^{4} \int_{\gamma_i} \frac{1}{k(z)} dz = \int_{\gamma} \frac{1}{k(z)} dz. \tag{23}$$

We set $I_i(\epsilon) = \int_{\gamma_i} \frac{1}{k(z)} dz$, $i = 1, 2, 3, 4$. Then

$$I_1(\epsilon) = \int_{\gamma_1} \frac{1}{k(z)} dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda)} d\lambda = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} d\lambda,$$
$$I_2(\epsilon) = \int_{\gamma_2} \frac{1}{k(z)} dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda + \sqrt{-1}\epsilon)} d\lambda,$$
$$I_3(\epsilon) = \int_{\gamma_3} \frac{1}{k(z)} dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} d\lambda,$$
$$I_4(\epsilon) = \int_{\gamma_4} \frac{1}{k(z)} dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda + \sqrt{-1}\epsilon)} d\lambda.$$
Since $k(z)$ is real for $z \in \mathbb{R}$, $k(z) = k(\bar{z})$ for $z, \bar{z} \in U$. Thus

$$I_1(\epsilon) + I_3(\epsilon) = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \left\{ \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} - \frac{1}{k(\lambda + \sqrt{-1}\epsilon)} \right\} d\lambda$$

$$= 2\sqrt{-1} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} d\lambda. \tag{24}$$

We also note that

$$|I_2(\epsilon)| = \left| \int_{\gamma_2} \frac{1}{k(z)} dz \right| = \left| -\sqrt{-1} \int_{-\epsilon}^{\epsilon} \frac{1}{k(\lambda_0 - \delta + it)} dt \right|$$

$$\leq \int_{-\epsilon}^{\epsilon} \left| \frac{1}{k(\lambda_0 - \delta + \sqrt{-1}t)} \right| dt$$

$$\leq 2\epsilon M, \tag{26}$$

and in the same way as above,

$$|I_4(\epsilon)| \leq 2\epsilon M. \tag{27}$$

On the other hand, by virtue of Cauchy's integral theorem,

$$\int_{\gamma} \frac{1}{k(z)} dz = \int_{\gamma} \frac{1}{(z - \lambda_0)k'(\lambda_0)} dz + \int_{\gamma} \varphi(z) dz$$

$$= \int_{\gamma} \frac{1}{(z - \lambda_0)k'(\lambda_0)} dz = 2\pi \sqrt{-1}/k'(\lambda_0). \tag{28}$$

Combining this with (23), (24), (26), (27) and (28), we arrive at

$$\lim_{\epsilon \downarrow 0} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \Re \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} d\lambda = \frac{\pi}{k'(\lambda_0)}. \tag{29}$$

References


Elementary solutions of Bessel processes with boundary conditions

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We consider elementary solutions of 2 dimensional Bessel processes on finite or infinite intervals, where some boundary conditions are posed at finite end points. The elementary solution of Bessel process on the interval $(0, \infty)$ is well known. We give explicit spectral representations of elementary solutions of Bessel processes on intervals $(0, a)$, $(a, b)$, and $(b, \infty)$ with various boundary conditions at $a$ or $b$, where $0 < a < b < \infty$. Our results imply that the elementary solution corresponding to $(b, \infty)$ converges to that corresponding to $(0, \infty)$ as $b \to 0$, and that corresponding to $(a, b)$ converges to that corresponding to $(0, b)$ as $a \to 0$. 