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Elementary solutions of Bessel processes with boundary conditions

TAKEMURA Tomoko *

1 Introduction

Let \( L_* \) be the diffusion operator on an interval \( I = (l_1, l_2), 0 \leq l_1 < l_2 \leq \infty \), defined by

\[
L_* = \frac{1}{2} \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right),
\]

which is the generator of a Bessel process. It is well known that, if \( l_1 = 0 \) [resp. \( l_2 = \infty \)], it is entrance [resp. natural] in the sense of Feller\[1\]. If \( 0 < l_i < \infty \) (\( i = 1, 2 \)), then \( l_i \) is regular in the same sense as above. Therefore a boundary condition must be posed at \( l_i \). In this paper we consider three kinds of boundary conditions, that is, absorbing, reflecting, and elastic boundary conditions.

Let \( p_{\alpha_1, \alpha_2}^{(l_1, l_2)}(t, x, y) \) be the elementary solution of the equation

\[
\frac{\partial u(t, x)}{\partial t} = L_* u(t, x), \quad t > 0, \ x \in I,
\]

in the sense of McKean\[5\], where \( \alpha_i \in \{ A, R, L, E, N \}, i = 1, 2 \), and \( \alpha_1 \) and \( \alpha_2 \) denote state of boundaries \( l_1 \) and \( l_2 \), respectively, that is, \( \alpha_i = A \) means that \( l_i \) is regular and absorbing, \( \alpha_i = R \) means that \( l_i \) is regular and reflecting, \( \alpha_i = L \) means that \( l_i \) is regular and elastic, and further \( \alpha_i = E \) means that \( l_i \) is entrance (and hence \( i = 1 \) and \( l_1 = 0 \)), \( \alpha_i = N \) means that \( l_i \) is natural (and hence \( i = 2 \) and \( l_2 = \infty \)).

The aim of this paper is to give explicit spectral representations of \( p_{\alpha_1, \alpha_2}^{(l_1, l_2)}(t, x, y) \) for \( \alpha_i \in \{ A, R, L, E, N \}, i = 1, 2 \). In the next section we state our results. In order to show them, we follow the same argument as in \[4\], \[5\], \[6\], \[9\], \[10\], which is summarized in Sect. 3. Sect. 4 is devoted to their proofs.

2 Main results

Let \( I = (l, \infty) \), \( dm \) be a nonnegative Borel measure on \( I \), which is finite on each compact set of \( I \), and \( s \) be a continuous increasing function on \( I \). For an arbitrarily fixed point \( c \in I \), we set \( m(x) = m((c, x]), m(l) = m((+ \infty, 0], m(r) = m([r, + \infty]) \in [0, \infty], s(l) = s(1+ \infty) \in (-\infty, 0), s(r) = s(1-) \in (-\infty, \infty) \). For a function \( f \) on \( I \), \( D_s f \) stands for the right derivative of \( f \) with respect to \( s \) if it exists, that is, \( D_s f(x) = \lim_{\epsilon \to 0} \{ f(x + \epsilon) - f(x) \}/\{ s(x + \epsilon) - s(x) \} \). We set

\( I^* = I \cup \{ x; x = l \text{ with } |s(l)| + |m(l)| < \infty \text{ or } x = r \text{ with } |s(r)| + |m(r)| < \infty \}. \)
\[ I_*(m) = \{ x \in I; m(x_1) < m(x_2) \text{ for } l < \forall x_1 < x < \forall x_2 < r \}, \]
\[ I_+ (m) = I_*(m) \cup \{ x; x = l \text{ with } |s(l)| + |m(l)| < \infty \]
\[ \text{or } x = r \text{ with } |s(r)| + |m(r)| < \infty \}. \]

Throughout this paper we assume that
\[ I(m) \neq \emptyset, \quad \inf I_*(m) = l, \quad \sup I_*(m) = r. \]

Let \( C_b(E) \) be the set of all bounded continuous function on \( E \), where \( E \) is a Borel set. Let \( D(L) \) be the space of all functions \( u \in C_b(I^*) \) for which there exists a function \( f \in C_b(I^*(m)) \) satisfying the following conditions.

\((L.1)\) There are two constants \( A_1, A_2 \) such that
\[ u(x) = A_1 + A_2 \{ s(x) - s(c) \} + \int_{(c,x)} \{ s(x) - s(y) \} f(y) \, dm(y), \quad x \in I. \]

\((L.2)\) If \( |s(l)| + |m(l)| < \infty \), then
\[ \theta_1 u(l) - \theta_2 D_s u(l+ - 0, \quad \theta_1 u(l) + \theta_2 D_s u(l-) = 0, \]
where \( \theta_i, \ i = 1, 2, \) are nonnegative numbers satisfying \( \theta_1 + \theta_2 = 1 \).

\((L.3)\) If \( |s(r)| + |m(r)| < \infty \), then
\[ \theta_1 u(r) + \theta_2 D_s u(r- - 0, \quad \theta_1 u(r) + \theta_2 D_s u(r+) = 0, \]
where \( \theta_i, \ i = 1, 2, \) are nonnegative numbers satisfying \( \theta_1 + \theta_2 = 1 \).

\((3)\) with \( \theta_2 = 0 \) [resp. \( (4) \) with \( \theta_2 = 0 \)] is called the absorbing boundary condition at \( l \) [resp. \( r \)]. \((3)\) with \( \theta_1 > 0 \) and \( \theta_2 > 0 \) [resp. \( (4) \) with \( \theta_1 > 0 \) and \( \theta_2 > 0 \)] is called the elastic boundary condition at \( l \) [resp. \( r \)]. \((3)\) with \( \theta_1 = 0 \) [resp. \( (4) \) with \( \theta_1 = 0 \)] is called the reflecting boundary condition at \( l \) [resp. \( r \)]. \((3)\) with \( |s(l)| + |m(l)| < \infty \) [resp. \( |s(r)| + |m(r)| < \infty \)] implies that \( l \) [resp. \( r \)] is regular in the sense of Feller [1]. In the next section we precisely state the classification of boundaries due to Feller. The operator \( L \) is defined by the mapping from \( u \in D(L) \) to \( f \in C_b(I_*(m)) \). \( m \) and \( s \) are called the speed measure and the scale function for \( C \), respectively.

Let \( \mathbb{D} = \{ X(t); t \geq 0, P_x; x \in I_*, \} \) be a generalized diffusion process whose generator is given by \( L \). Then it is known that there exists the transition probability density \( p(t,x,y) \) such that
\[ P_x(X(t) \in E) = \int_E p(t,x,y) \, dm(y), \]
for \( t > 0, \ x \in I_*(m), \ E \in B(I_*(m)), \) where \( B(E) \) stands for the set of all Borel sets of \( E \) ([3]). The function \( p(t,x,y) \) is the elementary solution of the equation (2) with \( L \) in place of \( L_* \). Note that \( p(t,x,y) = p(t,y,x) \) and \( p(t,x,y) \) is positive and continuous for \( t > 0, \ x, y \in I \).
Now we state our result. We go back to the diffusion operator $L_*$ on $(l_1, l_2)$ defined by (1). It is easy to see that the scale function $s_*$ and the speed measure $m_*$ for $L_*$ are given by

$$s_*(x) = C^{-1} \log x, \quad dm_*(x) = 2C x \, dx,$$

where $C$ is a positive constant. We may set $C = 1$ without loss of generality. Our results are represented by means of Bessel functions $J_\nu(x)$, $N_\nu(x)$ and modified Bessel functions $I_\nu, K_\nu$ with $\nu = 0, 1$, which are given as follows. For $x > 0$,

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)};$$

$$N_\nu(x) = \frac{2}{\pi} J_\nu(x) \left(\gamma + \log \frac{x}{2}\right)$$

$$- \frac{1}{\pi} \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(\nu + n + 1)} \left[ \sum_{m=1}^n \frac{1}{m} + \sum_{m=1}^{n+1} \frac{1}{m} \right]$$

$$- \frac{1}{\pi} \left(\frac{x}{2}\right)^{\nu-1} \sum_{r=0}^{\nu-1} \frac{(\nu - r - 1)!}{r!} \left(\frac{x}{2}\right)^{2r};$$

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(x/2)^{2n}}{n! \Gamma(\nu + n + 1)};$$

$$K_\nu(x) = (-1)^{\nu+1} I_\nu(x) \left(\gamma + \log \frac{x}{2}\right)$$

$$+ \frac{(-1)^\nu}{2} \sum_{n=0}^\infty \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)} \left[ \sum_{m=1}^n \frac{1}{m} + \sum_{m=1}^{n+1} \frac{1}{m} \right]$$

$$+ \frac{1}{2} \sum_{r=0}^{\nu-1} (-1)^r \frac{\nu - r - 1)!}{r!} \left(\frac{x}{2}\right)^{2r} \nu,$$

where $\gamma$ is Euler's constant and $\gamma = 0.57721 \ldots$.

It is well known that the elementary solution $p_0^E(t, x, y)$ is given by

$$p_0^E(t, x, y) = \frac{1}{2t} e^{-\frac{(x^2 + y^2)}{2t}} I_0(xy/t),$$

$$= \frac{1}{2} \int_0^\infty e^{-\lambda t} J_0(\sqrt{2\lambda} x) J_0(\sqrt{2\lambda y}) \, d\lambda.$$

First we consider the case that $l_1 = 0$ and $l_2 = a \in (0, \infty)$.

**Proposition 1.** Let $l_1 = 0$ and $l_2 = a \in (0, \infty)$. Assume the boundary condition (4) with $r, \theta_1$ and $\theta_2$ replaced by $a, \theta_1$ and $\theta_2$, respectively. Then $p^E(t, x, y)$ is given by

$$p^E(t, x, y) = \frac{\pi^2}{4} \sum_{\zeta \in k(\theta_1, \theta_2)} \frac{\zeta^2 \{ \theta_1 N_0(\zeta a) - \zeta a \theta_2 N_1(\zeta a) \}^2}{\theta_1^2 + \zeta^2 a^2 \theta_2^2} e^{-\zeta^2 t/2} J_0(\zeta x) J_0(\zeta y),$$

(6)
where $\alpha = A$ or $L$ or $R$; $\theta_2 = 0$ if $\alpha = A$; $\theta_1 = 0$ if $\alpha = R$; $\mathcal{K}(\theta_1, \theta_2) = \{\zeta \in [0, \infty); \theta_1 J_0(\zeta a + \zeta \theta_2 J_1(\zeta a))\}$.

We note the formula (6) with $\alpha = 1$ and $\theta_2 = 0$ is given in Sect. 5 of [2].

We next consider the case that $l_1 = a \in (0, \infty)$ and $l_2 = \infty$. In this case the spectrum only consists of continuous one. Let

$$
F(\lambda; \xi; \eta) = N_0(\sqrt{2\lambda \xi})J_1(\sqrt{2\lambda \eta}) - J_0(\sqrt{2\lambda \xi})N_1(\sqrt{2\lambda \eta}),
$$

$$
G_\nu(\lambda; \xi; \eta) = J_\nu(\sqrt{2\lambda \xi})N_\nu(\sqrt{2\lambda \eta}) - N_\nu(\sqrt{2\lambda \xi})J_\nu(\sqrt{2\lambda \eta}),
$$

for $\lambda, \xi, \eta \in (0, \infty)$ and $\nu = 0, 1$.

**PROPOSITION 2.** Let $l_1 = a \in (0, \infty)$ and $l_2 = \infty$. Assume the boundary condition (3) with $a$, $\theta_1$, and $\theta_2$ in place of $1$, $\theta_1'$, and $\theta_2'$, respectively. Then $p_{\alpha, \infty}^N(t, x, y)$ is given by

$$
p_{\alpha, \infty}^N(t, x, y) = \int_0^\infty e^{-\lambda t} \varphi_{(\alpha, \infty)}^N(x, \lambda) \varphi_{(\alpha, \infty)}^N(y, \lambda) \sigma_{(\alpha, \infty)}^N(\lambda) d\lambda,
$$

where $\alpha = A$ or $L$ or $R$; $\theta_2 = 0$ if $\alpha = A$; $\theta_1 = 0$ if $\alpha = R$.

**PROPOSITION 3.** Let $l_1 = a \in (0, \infty)$ and $l_2 = b \in (0, \infty)$. Assume that the boundary condition (3) with $a$, $\theta_1^a$ and $\theta_2^a$ in place of $1$, $\theta_1'$ and $\theta_2'$ and the boundary condition (4) with $b$, $\theta_1^b$ and $\theta_2^b$ in place of $r$, $\theta_1'$ and $\theta_2'$. Then $p_{(a, b)}^\beta(t, x, y)$ is given by

$$
p_{(a, b)}^\beta(t, x, y) = \sum_{\zeta \in \mathcal{K}(\theta_1^a, \theta_2^a, \theta_1^b, \theta_2^b)} e^{-\zeta t} \varphi_{(a, b)}^{\alpha\beta}(x, \zeta) \varphi_{(a, b)}^{\alpha\beta}(y, \zeta) \sigma_{(a, b)}^\alpha(\zeta),
$$

where

$$
\varphi_{(a, b)}^{\alpha\beta}(x, \zeta) = \frac{\theta_2^a}{\theta_1^a} \frac{\sqrt{2\zeta \pi}}{2} F(\zeta; x; a) + \frac{\pi}{2} G_0(\zeta; a; x), \quad \text{if } \alpha = A \text{ or } L,
$$

$$
\varphi_{(a, b)}^{R\beta}(x, \zeta) = \frac{\sqrt{2\zeta \pi}}{2} F(\zeta; x; a),
$$

$$
\sigma_{(a, b)}^\alpha(\zeta) = \left[ \frac{1}{2\zeta} - \left( \frac{\theta_2^a}{\theta_1^a} \right)^2 \right]^{-1} + \left\{ \frac{\theta_1^b}{2\lambda a \theta_1^a \theta_1^b} \left( \theta_1^b \right)^2 b \theta_1^b F(\zeta; a; b) - \theta_2^b \sqrt{2\zeta \pi a G_1(\zeta; a; b)} \right\}^{-1} \text{if } \alpha = A \text{ or } L.
\[ c_{(a,b)}^{R\beta}(\zeta) = \left[ -a^2 + ab \frac{\theta_1^2 \pi G_1(\zeta; a; b) + \theta_2^2 \sqrt{2\zeta} b F(\zeta; a; b)}{\theta_1^2 \pi G_0(\zeta; a; b) + \theta_2^2 \sqrt{2\zeta} ab F(\zeta; a; b)} \right]^{-1} \]

\[ \mathbb{L}(\theta_1^a, \theta_2^b) = \left\{ \zeta \in [0, \infty); \theta_2 \left[ \theta_1^a \sqrt{2\zeta} a F(\zeta; b; a) - 2 \theta_2^b ab \zeta G_1(\zeta; a; b) \right] \right\} = -\theta_1^a \left[ \theta_1^a G_0(\zeta; a; b) + \theta_2^b \sqrt{2\zeta} b F(\zeta; a; b) \right] \}

The following result is a direct consequence of Propositions 1, 2 and 3.

**COROLLARY 4.** The following formulae hold true.

\[ \lim_{\alpha \to 0} p_{(a, \infty)}^{AN}(t, x, y) = p_{(0, \infty)}^{EN}(t, x, y) \]

\[ \lim_{\alpha \to 0} p_{(a,b)}^{A\beta}(t, x, y) = p_{(0,b)}^{E\beta}(t, x, y) \]

where \( a, \beta \in \{A, L, R\} \).

This result is also obtained by using a convergence theorem on a sequence of elementary solutions due to Ogura [7].

### 3 Preliminaries

In this section we summarize some facts on elementary solutions, which are discussed in [3], [4], [5], [6], [9], [10], etc. Let \( \mathcal{L} \) be the operator defined at the beginning of the preceding section. Let \( m \) and \( s \) be the speed measure and the scale function for \( \mathcal{L} \), respectively. We introduce the following two quantities.

\[ J^l_{\mu, \nu} = \int_{(l, c]} dv(x) \int_{[x, c]} d\mu(y), \]

\[ J^r_{\mu, \nu} = \int_{[c, r)} dv(x) \int_{[x, c]} d\mu(y), \]

where \( d\mu \) and \( dv \) are Borel measure on \( I = (l, r) \) and \( c \in I_*(m) \). We note that \( J^l_{\mu, \nu} \) [resp. \( J^r_{\mu, \nu} \)] is finite for some \( c \in I_*(m) \) if and only if \( J^l_{\mu, \nu} \) [resp. \( J^r_{\mu, \nu} \)] is finite for each \( c \in I_*(m) \). For \( a = l, r, a \) is called to be

- regular if \( J^a_{m,a} < \infty, J^a_{s,a} < \infty \),
- exist if \( J^a_{m,a} = \infty, J^a_{s,a} < \infty \),
- entrance if \( J^a_{m,a} < \infty, J^a_{s,a} = \infty \),
- natural if \( J^a_{m,a} = \infty, J^a_{s,a} = \infty \).

It is easy to see that, for each \( a = l, r \),

- \( |m(a)| + |s(a)| < \infty \) if \( a \) is regular,
- \( |m(a)| = \infty, |s(a)| < \infty \) if \( a \) is exist,
- \( |m(a)| < \infty, |s(a)| = \infty \) if \( a \) is entrance,
- \( |m(a)| = \infty, |s(a)| = \infty \) if \( a \) is natural.
In the rest of this section we assume that $l$ is regular. We define the elementary solution $p(t, x, y)$ of the generalized diffusion equation (2) with $L$ replaced by $L_1$.

Here and hereafter we use the conventions $1/\infty = 0$ and $\pm a/0 = \pm \infty$ for a positive constant $a$. We put

\[ \tilde{t} = l - \theta_2^1/\theta_1^1, \]
\[ \tilde{r} = \begin{cases} r & \text{if } |m(r)| + |s(r)| = \infty, \\ r + \theta_2^1/\theta_1^1 & \text{if } |m(r)| + |s(r)| < \infty, \\ -\infty, & \text{if } \tilde{r} < l, \\ m(l), & l \leq x < l, \\ m(x), & l \leq x < r, \\ m(r), & r \leq x < \tilde{r}, \quad \text{if } |m(r)| + |s(r)| < \infty, \\ \infty, & \tilde{r} \leq x, \end{cases} \]

\[ S_m = (\tilde{t}, \tilde{r}), \]

\[ s(x) = \begin{cases} x - l + s(l), & \tilde{r} \leq x < l, \\ s(x), & l \leq x < r, \\ x - r + s(r) & r \leq x < \tilde{r}, \quad \text{if } |m(r)| + |s(r)| < \infty. \end{cases} \]

### 3.1 The case that $\tilde{t} < -\infty$.

We note that $\tilde{t} < -\infty$ implies that $l$ is absorbing or elastic with $\theta_1^1 > 0$. Let $\varphi_i(x, \alpha), i = 1, 2, \alpha \in \mathbb{C}$, be the solutions of the integral equations

\[ \varphi_1(x, \alpha) = 1 + \alpha \int_{(l,x]} \{s(x) - s(y)\} \varphi_1(y, \alpha) \, dm(y), \quad x \in S_m, \quad (11) \]
\[ \varphi_2(x, \alpha) = s(x) - s(l) + \alpha \{s(x) - s(y)\} \varphi_2(y, \alpha) \, dm(y), \quad x \in S_m. \quad (12) \]

Then for each $\alpha > 0$, there exists the limit

\[ k(\alpha) = \lim_{x \to \tilde{t}} (\alpha) = \varphi_2(x, \alpha)/\varphi_1(x, \alpha). \quad (13) \]

The function $k(\alpha)$ can be analytically continued to $\mathbb{C} \setminus (-\infty, 0]$. The spectral measure $\sigma_0$ is defined by

\[ \sigma_0([\lambda_1, \lambda_2]) = -\lim_{(0, \pi} \int_{\lambda_1}^{\lambda_2} \frac{1}{k(-\lambda - \sqrt{-1}\epsilon)} \, d\lambda, \]

for all continuity points $\lambda_1$ and $\lambda_2$ of $\sigma_0 (\lambda_1 < \lambda_2)$. We define the elementary solution of the generalized diffusion equation (2) by

\[ p(t, x, y) = \int_{(0, \infty)} e^{-\lambda t} \varphi_2(x, -\lambda) \varphi_2(y, -\lambda) \sigma_0(\lambda) \, d\lambda, \quad (14) \]

for $t > 0, x, y \in S_m$. 

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---
3.2 The case that $l = -\infty$.

This is the case that $l$ is reflecting. Let $\psi_i(x, \alpha), i = 1, 2, \alpha \in \mathbb{C}$, be the solutions of the integral equations

\[ \psi_1(x, \alpha) = 1 + \alpha \int_{(x,x]} \{s(x) - s(y)\} \psi_1(y, \alpha) \, dm(y), \quad x \in S_m, \quad (15) \]

\[ \psi_2(x, \alpha) = s(x) - s(l) + \alpha \int_{(x,x]} \{s(x) - s(y)\} \psi_2(y, \lambda) \, dm(y), \quad x \in S_m. \quad (16) \]

Then for each $\alpha > 0$, there exists the limit

\[ h(\alpha) = \lim_{x \to x} \frac{\psi_2(x, \alpha)}{\psi_1(x, \alpha)}. \quad (17) \]

The function $h(\alpha)$ can be analytically continued to $\mathbb{C} \setminus (-\infty, 0]$. The spectral measure $\sigma^0$ is defined by

\[ \sigma^0([\lambda_1, \lambda_2]) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Re h(-\lambda - \sqrt{-1}\epsilon) \, d\lambda, \]

for all continuity points $\lambda_1$ and $\lambda_2$ of $\sigma^0 (\lambda_1 < \lambda_2)$. We define the elementary solution of the generalized diffusion equation (2) by

\[ p(t, x, y) = \int_{[0,\infty)} e^{-\lambda t} \psi_1(x, -\lambda) \psi_1(y, -\lambda) \sigma^0(d\lambda), \quad (18) \]

for $t > 0$, $x, y \in S_m$.

4 Spectral representations of elementary solutions

In this section we prove Propositions 1, 2 and 3. Throughout this section, we set

\[ \Phi_a(x, \alpha) = \sqrt{2\alpha \pi} \{ K_1(\sqrt{2\alpha}x)J_0(\sqrt{2\alpha}x) + I_1(\sqrt{2\alpha}x)K_0(\sqrt{2\alpha}x) \}, \quad (19) \]

\[ \Phi_a(x, -\alpha) = \frac{\sqrt{2\alpha \pi}}{2} F(\alpha; x; a), \quad (20) \]

\[ \Psi_a(x, \alpha) = K_0(\sqrt{2\alpha}x)I_0(\sqrt{2\alpha}x) - I_0(\sqrt{2\alpha}x)K_0(\sqrt{2\alpha}x), \quad (21) \]

\[ \Psi_a(x, -\alpha) = \frac{\pi}{2} G_0(\alpha; a; x), \quad (22) \]

for $x > 0$ and $\alpha > 0$, where $a$ is a positive number. It is easy to see that

\[ \Phi_a(a, \alpha) = 1, \quad \frac{\partial}{\partial x} \Phi_a(x, \alpha)|_{x=a} = 0, \]

\[ \Psi_a(a, \alpha) = 0, \quad \frac{\partial}{\partial x} \Psi_a(x, \alpha)|_{x=a} = \frac{1}{a}. \]
4.1 The Case that \( l_1 = 0 \) and \( l_2 = a \in (0, \infty) \).

We prove Proposition 1. Then \( a \) is regular, and satisfies the boundary condition (4) with \( r = a, \theta_1' = \theta_1 \) and \( \theta_2' = \theta_2 \). We set

\[
\tilde{r} = a + \frac{\theta_2/\theta_1}{}, \quad s(x) = \begin{cases} \log x, & 0 < x < a, \\ x - a + \log a, & a \leq x \leq \tilde{r}. \end{cases}
\]

We apply the argument in the preceding section exchanging the role of \( l_1 \) and \( l_2 \).

Assume that \( a \) is absorbing or elastic. Then \( \varphi_1(x, \alpha), \varphi_2(x, \alpha), \) and \( k(\alpha) \) corresponding to (11), (12) and (13) are given as follows.

\[
\varphi_1(x, \alpha) = \begin{cases} \Phi_\alpha(x, \alpha), & 0 < x \leq a, \\ 1, & a < x < \tilde{r}, \end{cases}
\]

\[
\varphi_2(x, \alpha) = \begin{cases} -\Psi_\alpha(x, \alpha) + (\theta_2/\theta_1)\Phi_\alpha(x, \alpha), & 0 < x \leq a, \\ \tilde{r} - x, & a < x \leq \tilde{r}, \end{cases}
\]

\[
k(\alpha) = \lim_{x \to 0} \frac{\varphi_2(x, \alpha)}{\varphi_1(x, \alpha)} = \frac{\theta_2}{\theta_1} + \frac{1}{\sqrt{2\alpha a}} \frac{I_0(\sqrt{2\alpha a})}{I_1(\sqrt{2\alpha a})}, \quad \alpha > 0.
\]

Let us fix \( \zeta \in \Lambda(\theta_1, \theta_2) := \{ \zeta \in (0, \infty) \mid \theta_1\nu_0(\sqrt{2\zeta a}) = \theta_2\nu_1(\sqrt{2\zeta a}) \} \). Then, by means of Proposition A, there exists a positive \( \delta > 0 \) such that

\[
\sigma_0(1, 1) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \left( \frac{1}{k(-\lambda - \sqrt{-1}\epsilon)} \right) d\lambda = \frac{2\zeta}{1 + 2(\theta_2/\theta_1)^2 \alpha^2 \zeta},
\]

for \( \zeta - \delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta \). By means of (20) and (22), (14) is reduced to

\[
\sum_{\zeta \in \Lambda(\theta_1, \theta_2)} e^{-\zeta t} \varphi_2(x, -\zeta) \varphi_a(y, -\zeta) \mathfrak{J}/\{1 + 2(\theta_2/\theta_1)^2 \alpha^2 \zeta\}
\]

\[
= \frac{\pi^2}{4} \sum\limits_{\zeta \in \mathbb{J}(0)} \zeta^2 e^{-\zeta^2 t/2} \nu_0(\zeta^2 a^2) J_0(\zeta x) J_0(\zeta y),
\]

which shows (6) for \( \alpha = A \) or \( L \).

Next assume that \( a \) is reflecting. Then \( \psi_1, \psi_2, \) and \( h \) corresponding to (15), (16) and (17) are given as follows.

\[
\psi_1(x, \alpha) = \Phi_\alpha(x, \alpha), \quad \psi_2(x, \alpha) = -\Psi_\alpha(x, \alpha), \quad 0 < x \leq a,
\]

\[
h(\alpha) = \lim_{x \to 0} \frac{\varphi_2(x, \alpha)}{\varphi_1(x, \alpha)} = \frac{1}{\sqrt{2\alpha a}} \frac{I_0(\sqrt{2\alpha a})}{I_1(\sqrt{2\alpha a})}.
\]

Let us fix \( \zeta \in [0, \infty) \) such that \( \sqrt{2\zeta J_1(\sqrt{2\zeta a})} = 0 \). By means of Proposition A, there exists a positive \( \delta > 0 \) and

\[
\sigma^0(\lambda_1, \lambda_2) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \left( \frac{1}{k(-\lambda - \sqrt{-1}\epsilon)} \right) d\lambda = \frac{1}{a^2},
\]
for $\zeta - \delta < \lambda_1 < \zeta < \lambda_2 < \zeta + \delta$. By means of (20) and (22), (18) is reduced to

$$
\sum_{\zeta; J_1(\sqrt{2\alpha A})=0} e^{-\zeta t} \psi_1(x, -\zeta) \psi(y, -\zeta)/\alpha^2 = \frac{\pi^2}{4} \sum_{\zeta; J_1(\zeta)=0} \zeta^2 e^{-\zeta t} N_1(\zeta A)^2 J_0(\zeta x) J_0(\zeta y),
$$

which shows (6) for $\alpha = R$.

4.2 The Case that $l_1 = a \in (0, \infty)$ and $l_2 = \infty$.

We prove Proposition 2. Then $a$ is regular and the boundary condition for (3) with $l = a$, $\theta_1 = \theta_1$ and $\theta_2 = \theta_2$. We set

$$
\tilde{l} = a - \theta_2/\theta_1,
$$

$$
\tilde{r} = \infty,
$$

$$
m(x) = \begin{cases} 0, & x \leq \tilde{l}, \\ a^2, & \tilde{l} < x < a, \\ x^2, & a \leq x, \end{cases}
$$

$$
s(x) = \begin{cases} x - a + \log a, & \tilde{l} < x \leq a, \\ \log x, & a < x. \end{cases}
$$

Assume that $a$ is absorbing or elastic. Then $\varphi_1, \varphi_2$ and $k$ define by (11), (12) and (13) are given as follows.

$$
\varphi_1(x, a) = \Phi_a(x, a), \quad \varphi_2(x, a) = \frac{\theta_2}{\theta_1} \Phi_a(x, a) + \Psi_a(x, a), \quad a \leq x < \infty,
$$

$$
k(a) = \lim_{x \to r} \varphi_2(x, a)/\varphi_1(x, a) = \frac{\theta_2}{\theta_1} + \frac{K_0(\sqrt{2\alpha a})}{\sqrt{2\alpha a} K_1(\sqrt{2\alpha a})}.
$$

Therefore

$$
\sigma_0([\lambda_1, \lambda_2]) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \left( \frac{1}{k(-\lambda - \sqrt{-1\epsilon})} \right) d\lambda
$$

$$
= \frac{2\theta_1^2}{\pi^2} \int_{\lambda_1}^{\lambda_2} \left\{ \theta_1 N_0(\sqrt{2\lambda a}) + \sqrt{2\lambda a} J_0((\sqrt{2\lambda a}))^2 \right\} d\lambda,
$$

for $0 < \lambda_1 < \lambda_2 < \infty$. Noting (20) and (22) we obtain (9) for $\alpha = A$ or $L$.

We next assume that $a$ is reflecting. The $\psi_1, \psi_2$, and $h$ defined by (15), (16) and (17) are given as follows.

$$
\psi_1(x, a) = \Phi_a(x, a), \quad \psi_2(x, a) = \Psi_a(x, a), \quad a \leq x < \infty,
$$

$$
h(a) = \frac{K_0(\sqrt{2\alpha a})}{\sqrt{2\alpha a} K_1(\sqrt{2\alpha a})}.
$$

Therefore

$$
\sigma^0([\lambda_1, \lambda_2]) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im h(-\lambda - \sqrt{-1\epsilon}) d\lambda
$$

$$
= \int_{\lambda_1}^{\lambda_2} \frac{1}{a^2 \lambda \pi^2} \frac{1}{J_1(\sqrt{2\alpha a})^2 + N_1(\sqrt{2\alpha a})^2} d\lambda.
$$

Combining this with (22), we obtain (9) for $\alpha = R$. 

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4.3 The Case that $l_1 = a \in (0, \infty)$ and $l_2 = b \in (0, \infty)$.

We prove Proposition 3. Since $a$ and $b$ are regular, the boundary condition (3) is posed at $a$, and (4) with $\tau = b$ is posed at $b$. We set

$$m(x) = \begin{cases} 
-\infty, & x \leq \tilde{l}, \\
a^2, & \tilde{l} \leq x < a, \\
x^2, & a \leq x < \hat{r}, \\
b^2, & b \leq x < \hat{r}, \\
\infty, & \hat{r} \leq x,
\end{cases}$$

$$\phi(x) = \begin{cases} 
x - a + \log a, & \tilde{l} < x < a, \\
\log x, & a < x \leq b, \\
x - b + \log b, & b < x < \hat{r}.
\end{cases}$$

We assume that $a$ is absorbing or elastic. Then $\varphi_1(x, \alpha), \varphi_2(x, \alpha)$ and $k(\alpha)$ defined by (11), (12) and (13) are given as follows.

$$\varphi_1(x, \alpha) = \begin{cases} 
1, & \tilde{l} \leq x < a, \\
\Phi_\alpha(x, \alpha), & a \leq x \leq b, \\
\Phi_\alpha(b, \alpha) + (x - b)D_s\Phi_\alpha(b, \alpha), & b < x < \hat{r},
\end{cases}$$

$$\varphi_2(x, \alpha) = \begin{cases} 
x - \tilde{l}, & \tilde{l} \leq x < a, \\
Q_\alpha(x, \alpha) := (\theta^2_2/\theta^2_1)\Phi_\alpha(x, \alpha) + \Psi_\alpha(x, \alpha), & a \leq x < b, \\
Q_\alpha(b, \alpha) + (x - b)D_sQ_\alpha(b, \alpha), & b < x < \hat{r},
\end{cases}$$

$$k(\alpha) = \frac{\theta^2_2}{\theta^2_1} + \frac{\theta^2_1\Psi_\alpha(b, \alpha) + \theta^2_2D_s\Psi_\alpha(b, \alpha)}{\theta^2_1\Phi_\alpha(b, \alpha) + \theta^2_2D_s\Phi_\alpha(b, \alpha)}.$$
Let us fix $\zeta \in \Lambda(\theta_1^0, \theta_2^0) := \{\zeta \in (0, \infty); \theta_1^0 \sqrt{2} \zeta a F(\zeta; b; a) = 2\theta_2^0 b \zeta G_1(\zeta; a; b)\}$. Then, by means of Proposition A, there exists a positive $\delta > 0$ such that
\[
\sigma^0(\lambda_1, \lambda_2) = \left[-a^2 + ab \theta_1^0 \pi G_1(\zeta, a, b) + \theta_2^0 \sqrt{2} \zeta b F(\zeta, b, a)\right]^{-1}
\]
for $-\delta < \lambda_1 < \lambda_2 < \zeta + \delta$. By means of (20) and (22), we have (10) with $\alpha = R$.

5 Appendix

The following result is useful for calculations of spectrum. The proof is due to Dr. Sechiko Takahashi. We would like to thank her for suggesting improvements of our results.

**Proposition A** Let $k(z) : \mathbb{C} \to \mathbb{C}$ and $k(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Assume that there is a $\lambda_0 \in \mathbb{R}$ such that $k$ is analytic in a neighborhood of $\lambda_0$, $k(\lambda_0) = 0$, and it is a zero of order 1. Then there exists a positive $\delta$ such that
\[
\lim_{\epsilon \to 0} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} \, d\lambda = \frac{\pi}{k'(\lambda_0)}.
\]

**Proof.** By virtue of assumption of the proposition, there is a neighborhood $U$ of $\lambda_0$ such that $\lambda_0$ is a simple pole of $1/k$, and $\varphi(z) := 1/k(z) - 1/(z - \lambda_0)k'(\lambda_0)$ is analytic in $U$. Let us fix a $\delta > 0$ such that $[\lambda_0 - \delta, \lambda_0 + \delta] \subset U$. Further fix an $\epsilon_0 > 0$ such that
\[
L_1 := \{\lambda_0 - \delta + \sqrt{-1}\epsilon; -\epsilon_0 < \epsilon < \epsilon_0\} \subset U,
\]
\[
L_2 := \{\lambda_0 + \delta + \sqrt{-1}\epsilon; -\epsilon_0 < \epsilon < \epsilon_0\} \subset U.
\]
Since $1/k$ is analytic on $L_1$ and $L_2$, there is an $M > 0$ such that $|k(z)| \leq M$ for $z \in L_1 \cup L_2$.

For $0 < \epsilon < \epsilon_0$ and $0 < \rho < \min\{\delta, \epsilon\}$, we set
\[
\gamma_1 : z(\lambda) = \lambda + \sqrt{-1}\epsilon \quad (\lambda_0 + \delta \leq \lambda \geq \lambda_0 - \delta), \quad \gamma_2 : z(t) = \lambda - \delta + \sqrt{-1}t \quad (\epsilon \leq \epsilon \leq \epsilon_0),
\]
\[
\gamma_3 : z(\lambda) = \lambda - \sqrt{-1}\epsilon \quad (\lambda_0 - \delta \leq \lambda \leq \lambda_0 + \delta), \quad \gamma_4 : z(\lambda) = \lambda + \delta + \sqrt{-1}t \quad (-\epsilon \leq t \leq \epsilon),
\]
\[
\gamma : z = \rho e^{\sqrt{-1}t} \quad 0 \leq \theta \leq 2\pi \quad (\rho \leq \delta, \epsilon).
\]

By virtue of Cauchy’s integral theorem,
\[
\sum_{i=1}^{4} \int_{\gamma_i} \frac{1}{k(z)} \, dz = \int_{\gamma} \frac{1}{k(z)} \, dz.
\]

We set $I_i(\epsilon) = \int_{\gamma_i} \frac{1}{k(z)} \, dz$, $i = 1, 2, 3, 4$. Then
\[
I_1(\epsilon) = \int_{\gamma_1} \frac{1}{k(z)} \, dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda + \sqrt{-1}\epsilon)} \, d\lambda = -\int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} \, d\lambda,
\]
\[
I_3(\epsilon) = \int_{\gamma_3} \frac{1}{k(z)} \, dz = \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1}\epsilon)} \, d\lambda.
\]
Since \( k(z) \) is real for \( z \in \mathbb{R} \), \( k(z) = k(\overline{z}) \) for \( z, \overline{z} \in U \). Thus

\[
I_2(z) + I_3(z) = \int_{\lambda_0}^{\lambda_0 + \delta} \left\{ \frac{1}{k(\lambda - \sqrt{-1} \epsilon)} - \frac{1}{k(\lambda - \sqrt{-1} \epsilon)} \right\} d\lambda \tag{24}
\]

\[
= 2 \sqrt{-1} \int_{\lambda_0}^{\lambda_0 + \delta} \frac{1}{k(\lambda - \sqrt{-1} \epsilon)} d\lambda. \tag{25}
\]

We also note that

\[
|I_2(z)| = \left| \int_{\gamma_2} \frac{1}{k(z)} \, dz \right| = - \sqrt{-1} \int_{-\epsilon}^{\epsilon} \frac{1}{k(\lambda_0 - \delta + it)} \, dt
\]

\[
\leq \int_{-\epsilon}^{\epsilon} \left| \frac{1}{k(\lambda_0 - \delta + it)} \right| \, dt
\]

\[
\leq 2 \epsilon M, \tag{26}
\]

and in the same way as above,

\[
|I_4(z)| \leq 2 \epsilon M. \tag{27}
\]

On the other hand, by virtue of Cauchy's integral theorem,

\[
\int_{\gamma} \frac{1}{k(z)} \, dz = \int_{\gamma} \frac{1}{(z - \lambda_0) k'(\lambda_0)} \, dz + \int_{\gamma} \varphi(z) \, dz
\]

\[
= \int_{\gamma} \frac{1}{(z - \lambda_0) k'(\lambda_0)} \, dz = 2 \pi \sqrt{-1}/k'(\lambda_0). \tag{28}
\]

Combining this with (23), (24), (26), (27) and (28), we arrive at

\[
\lim_{\epsilon \to 0} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} \Im \left\{ \frac{1}{k(\lambda - \sqrt{-1} \epsilon)} \right\} d\lambda = \frac{\pi}{k'(\lambda_0)}.
\]

References


Elementary solutions of Bessel processes with boundary conditions

TAKEMURA Tomoko

We consider elementary solutions of 2 dimensional Bessel processes on finite or infinite intervals, where some boundary conditions are posed at finite end points. The elementary solution of Bessel process on the interval \((0, \infty)\) is well known. We give explicit spectral representations of elementary solutions of Bessel processes on intervals \((0, a)\), \((a, b)\), and \((b, \infty)\) with various boundary conditions at \(a\) or \(b\), where \(0 < a < b < \infty\). Our results imply that the elementary solution corresponding to \((b, \infty)\) converges to that corresponding to \((0, \infty)\) as \(b \to 0\), and that corresponding to \((a, b)\) converges to that corresponding to \((0, b)\) as \(a \to 0\).