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A condition for an infinitely generated Schottky group to be classical

Masahiko Taniguchi * † and Fumio Maitani ‡

1 Introduction and main theorems

In the case of finitely generated Kleinian groups, it was shown in [9] that there are Schottky groups that are not classical (that is, they cannot be defined by using circles); see also [7]. Examples of non-classical Schottky groups were constructed by various ways, for instance, in [5] and [17]. Cf. also [16]. On the other hand, the conditions for Schottky groups to be classical are known in [1] and [11].

In this paper, we introduce a new concept of infinitely generated Schottky groups. Recall that a definition of infinitely generated classical Schottky groups is given in the book [10]. See also [2].

Consider a set 
\[ \mathcal{C} = \{ C_j, C'_j \mid j \in \mathbb{N} \} \]
of countably infinite number of pairs of simple closed curves in \( \mathbb{C} \) such that not only these curves but also the interiors of them are mutually disjoint. Here, the interior of a simple closed curve \( C \) is the bounded connected component of \( \mathbb{C} - C \). The other component, together with \( \infty \), is called the exterior of \( C \).

We further assume that the exterior of \( C_j \) is mapped onto the interior of \( C'_j \) by a Möbius transformation \( g_j \) for every \( j \).

Definition 1. Let \( G \) be the group generated by all \( g_j \) defined as above. If \( G \) is discontinuous outside a compact totally disconnected set in \( \hat{\mathbb{C}} \), or equivalently, if the limit set \( \Lambda(G) \) of \( G \) is totally disconnected, then we call \( G \) an infinitely generated Schottky group with respect to the family \( \mathcal{C} \).

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Here, if all elements of $\mathcal{C}$ are circles, then we call $G$ an infinitely generated classical Schottky group.

Further, we impose two conditions on $\mathcal{C}$. In the sequel, we always assume that the family $\mathcal{C}$ is contained in a bounded set of $\mathbb{C}$.

**Definition 2** (Modified Maskit condition). For every element $C$ of $\mathcal{C}$, there is an annulus (i.e. a doubly connected domain) $W$ of constant modulus $d > 0$ such that $W$ separates $C$ from $\mathcal{C} \setminus \{C\}$.

Here the *modulus* $m(W)$ of $W$ is $\log(R/r)$ when we map $W$ conformally onto a concentric circle domain $\{r < |z| < R\}$.

**Definition 3.** We say that an annulus $W$ in $\mathcal{C}$ is *nested inside* another $W'$ if $W$ is contained in the bounded connected component of $\mathcal{C} \setminus W'$.

**Definition 4** (Tameness condition). There is an increasing sequence $\{N_j\}_{j=1}^\infty$ of positive integers such that, for every $N = N_j$, there is an annulus $W_N$ of constant modulus $d > 0$ which separates $\{C_j, C'_j \mid j = 1, \cdots, N\}$ from $\{C_j, C'_j \mid j \geq N + 1\}$ and is nested inside $W_{N_j - 1}$.

**Remark 1.** The tameness condition clearly implies that the cluster set of elements of $\mathcal{C}$ consists of a single point, say $z_\infty$.

**Theorem 1.** Let $G$ be the group generated by all $g_j$ defined as above. If $\mathcal{C}$ satisfies the modified Maskit condition and the tameness condition, then $G$ is an infinitely generated Schottky group with respect to $\mathcal{C}$.

For a more precise statement, see Lemma 5 below, which implies Theorem 1.

**Remark 2.** In the case of finitely generated Schottky groups, the modified Maskit condition trivially holds, and the tameness condition imposes nothing. Thus the above theorem reduces to the well-known properties of finitely generated Schottky groups. See for instance, [10] and [12].

Next, we introduce the concept of maximal symmetricity for Riemann surfaces uniformized by infinitely generated Schottky groups, which was considered in [11] for closed ones. For this purpose, consider the Riemann surface $R = (\hat{\mathbb{C}} - \Lambda(G))/G$. The simple loop on $R$ corresponding to $C_j$ is denoted by $L_j$ for every $j$. Set $\mathcal{L} = \{L_j \mid j \in \mathbb{N}\}$ and call it the *Schottky marking* of $R$ corresponding to $G$.

**Definition 5.** Let $f$ be an anti-conformal self-homeomorphism of $R$. we say that $f$ is *maximal with respect to $\mathcal{L}$* if the fixed point set of $f$ consists of
• a family \( G = \{ \gamma_j \mid j \in \mathbb{N} \} \) of simple closed curves such that every \( \gamma_j \) is freely homotopic to \( L_j \) on \( R \) for every \( j \), and

• another simple curve \( \gamma \).

We say that the Schottky marked Riemann surface \( R \) is \textit{maximally symmetric} if \( R \) has the maximal anti-conformal self-homeomorphism with respect to the Schottky marking \( \mathcal{L} \). The loops \( \gamma_j \) and \( \gamma \) are called the \textit{mirrors} of \( R \).

Now, we can state the main theorem of this paper, which is a natural generalization of a theorem of Maskit in [11].

**Theorem 2.** Let \( G \) be an infinitely generated Schottky group satisfying the modified Maskit condition and the tameness condition. Further suppose that the corresponding Schottky marked Riemann surface \( R \) is maximally symmetric. Then \( G \) is classical.

The proof will be given in the next section.

Finally, in Section 3, we discuss connection between the main theorem, Schottky uniformization, and the circle domain theorem, which is proved by Koebe [8] for finite-ply connected domains, and by He and Schramm [4] for ones with countably infinite number of boundary components.

## 2 Proofs of theorems

A crucial fact for the proof of Theorem 1 is the following famous classical result due to Ahlfors and Beurling.

**Proposition 3.** Let \( D \) be a domain in \( \hat{\mathbb{C}} \). Then every univalent holomorphic map of \( D \) into \( \hat{\mathbb{C}} \) is a Möbius transformation if and only if the complement of \( D \) in \( \hat{\mathbb{C}} \) belongs to the class \( N_D \), which is, by definition, equivalent to the condition that \( D \) belongs to the class \( O_{AD} \), i.e. that there are no non-constant holomorphic functions on \( D \) with finite Dirichlet energy.

In particular, if the complement \( E \) of \( D \) in \( \hat{\mathbb{C}} \) belongs to the class \( N_D \), then \( E \) is totally disconnected and every biholomorphic self-homeomorphism of \( D \) is a Möbius transformation.

Various practical tests for a compact set to belong to \( N_D \) have been considered. See for instance [14] and [15]. A famous one is the following formulation due to McMullen ([13]).

**Proposition 4** (Modulus test). Let \( \{ E_n \}_{n=1}^\infty \) be a sequence of a finite union of disjoint un-nested annuli (of finite moduli) such that
1. every component $W$ of $E_{n+1}$ is nested inside a component of $E_n$, and that

2. for every sequence of nested annuli $W_n$, which is a component of $E_n$, we have

$$\sum_{n=1}^{\infty} m(W_n) = +\infty.$$ 

Let $E'_n$ be the union of all bounded connected components of $\mathbb{C} - E_n$, and set

$$E = \bigcap_{n=1}^{\infty} E'_n.$$ 

Then $E$ is a totally disconnected compact set belonging to $N_D$.

Now, Theorem 1 follows from the following lemma.

**Lemma 5.** Let $D$ be the non-simply connected component of $\mathbb{C} - \bigcup_j (C_j \cup C'_j)$, the intersection of the exteriors of all elements in $\mathcal{C}$ deleted $z_\infty$, and let $E$ be the complement of the union

$$\Omega = \bigcup_{g \in G} \left( \overline{g(D)} - \{g(z_\infty)\} \right)$$

of the closures $\overline{g(D)}$ of all $g(D)$ deleted $g(z_\infty)$ with $g \in G$.

Then $E$ is totally disconnected and the limit set of $G$, and hence $G$ is discontinuous on $\Omega$. The Riemann surface $R = \Omega / G$ is obtained from $D$ by identifying every $C_j$ with $C'_j$ under the action of $g_j$. In other words, the domain $\Omega$ is the Schottky uniformization of $R$ with respect to $G$.

**Proof.** We construct a nested family $\{E_n\}$ such as desired in the modulus test from all images $g(W)$ by $g \in G$ of the annuli $W$ appeared in the modified Maskit condition and in the tameness condition.

First, let $\{N_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers such that, for every $N = N_j$, there is an annulus $W_N$ of constant modulus $d > 0$ which separates $\{C_k, C'_k \; | \; k = 1, \cdots, N\}$ from $\{C_k, C'_k \; | \; k \geq N + 1\}$ and is nested inside $W_{N_j - 1}$. Such a sequence exists by the tameness condition.

Next, set $\Omega_0 = D$, and $\Omega_n$ is the domain obtained from $\Omega_{n-1}$ by attaching copies of $D$ along all boundary components of $\Omega_{n-1}$ corresponding to $\{C_j, C'_j \; | \; j = 1, \cdots, N_n\}$ under the actions of elements of $G$ for every $n \in \mathbb{N}$.

Finally for every $n$, let $E_n$ be the finite union of
1. all annuli of modulus \(d > 0\) separating \(C\) from the all other boundary components of \(\Omega\) for boundary components \(C\) of \(\Omega\) corresponding to \(\{C_j, C_j' \mid j = 1, \ldots, N\}\) under the actions of elements of \(G\), which exist by the modified Maskit condition, and

2. all annuli in \(\Omega\) of the form \(g(W_{N_j})\) with \(g \in G\).

Then from the construction, it is clear that the sequence \(\{E_n \mid n \in \mathbb{N}\}\) satisfies the conditions 1) and 2) in the modulus test. And hence the union \(\Omega\) of all \(\Omega_n\) is the complement of a totally disconnected compact set \(E\) belonging to \(N_D\).

Since \(\Omega\), and hence also \(E\), is invariant under the action by \(G\), and since \(G\) is discontinuous exactly on \(\Omega\), we see that \(G\) is a Kleinian group having \(E\) as the limit set, which shows the assertion.

Now, to prove Theorem 2, assume that \(R\) has a maximal anti-conformal self-homeomorphism \(f\) with respect to the Schottky marking \(L\) of \(R\), and let \(\mathcal{G} = \{\gamma_j \mid j \in \mathbb{N}\}\) and \(\gamma\) be the mirrors of \(R\).

In the case of infinitely generated Schottky groups, \(\gamma\) may be non-compact. Also it is well-known that all \(\gamma_j\) and \(\gamma\) are geodesics with respect to the hyperbolic metric on \(R\). In particular, elements of \(\mathcal{G}\) are mutually disjoint and \(\gamma\) is disjoint from all \(\gamma_j\).

**Remark 3.** Firstly, by the modified Maskit condition for \(C\), the hyperbolic lengths of all \(\gamma_j\) are less than a common constant. This is the same for the geodesic, say \(g_{N_j}\), freely homotopic to the essential simple closed curve in every \(W_{N_j}\) appeared in the proof of Lemma 5.

Secondly, from the construction, there exists a set \(\Gamma = \{\tilde{\gamma}_j, \tilde{\gamma}_j' \mid j \in \mathbb{N}\}\)

of countable infinite number of pairs of simple closed curves in \(\hat{\mathbb{C}}\) such that \(\tilde{\gamma}_j\) and \(\tilde{\gamma}_j'\) are projected to \(\gamma_j\) on \(R\) and the exterior of \(\tilde{\gamma}_j\) is mapped by the Möbius transformation \(g_j\) onto the interior of \(\tilde{\gamma}_j'\) for every \(j\).

Finally, by the collar lemmas, we can see from above that \(\Gamma\) satisfies the modified Maskit condition and the tameness condition. Hence by Lemma 5, the intersection \(D'\) of the exteriors of all elements in \(\Gamma\) is also a fundamental domain for \(G\). In other words, \(G\) is an infinitely generated Schottky group with respect also to \(\Gamma\).

**Proof of Theorem 2.** For every \(j\), the maximally symmetric anti-conformal map \(f\) of \(R\) can be lifted to an anti-conformal homeomorphism \(\tau_j\) of \(\Omega = \hat{\mathbb{C}} - E\) which has \(\tilde{\gamma}_j\) as the fixed point set. Since \(E\) belongs to \(N_D\), \(\tau_j\) is a Möbius transformation pre-composed by the complex conjugate. Thus \(\tilde{\gamma}_j\) and hence also \(\tilde{\gamma}_j' = g_j(\tilde{\gamma}_j)\) should be a circle, which show the assertion. 

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3 A remark on Koebe circle theorem

An old conjecture stated in [6] and [11] is the following

**Conjecture 1** (Classical Schottky uniformizability). _Every closed Riemann surface is uniformizable by a finitely generated classical Schottky group._

The Maskit showed in [11] that this conjecture is true for every maximally symmetric closed Riemann surface. This conjecture has a closed connection with a so-called circle domain theorem of Koebe [8], which has been generalized by He and Schramm in [4].

It will have some meaning to include a proof which clarifies this connection in the case of finite-ply connected planar domains, which is essentially the same as the original one given by Koebe [8]. See also [3] and [11].

In the sequel, we say that a subdomain $D$ in $\hat{\mathbb{C}}$ is a _circle domain_ if every boundary component of $D$ is either a circle or a point.

**Theorem 6** (Koebe circle theorem [8]). _Every finite-ply connected planar domain $D$ can be mapped conformally onto a circle domain._

_Furthermore, two circle domains $D$ and $D'$ are conformally equivalent if and only if there is a Möbius transformation $T$ such that $T(D) = D'.$_

_Proof._ First, we may assume that all boundary components of $D$ are smooth simple closed curves, say $C_1, \cdots, C_n$ with $n > 1$. Then, attach an anti-conformal copy $D^*$ of $D$ to $D$ along a boundary component, say $C_1$, and we have a $(2n - 2)$-ply connected planar domain $\hat{D}$ with boundary components $\{C_j, C_j^* \mid j = 2, \cdots, n\}$.

Set $\Omega_0 = \hat{D}$. By definition, there is the anti-conformal homeomorphism of $D$ onto $D^*$, which gives not only the anti-conformal involution, say $\tau$, of $\Omega_0$ fixing $C_1$ point-wise, but also the anti-conformal homeomorphism, say $\tau_j^*$, of $\Omega_0$ onto the copy $\Omega^*_0,j$ of $\Omega_0$ attached along $C_j^*$ which fixes $C_j^*$ point-wise for every $j$. Set $g_j = \tau_j^* \circ \tau$, and we have a conformal map of $\Omega_0$ onto $\Omega^*_0,j$ which maps $C_j$ to $C_j^*$. Similarly, we obtain the anti-conformal homeomorphism, say $\tau_j$, of $\Omega_0$ onto the copy $\Omega_{0,j}$ of $\Omega_0$ attached along $C_j$ which fixes $C_j$ point-wise, and a conformal map $h_j = \tau_j \circ \tau : \Omega_0 \to \Omega_{0,j}$ for every $j$. Set $\Omega_1$ be the planar Riemann surface

$$\Omega_0 \cup \left( \bigcup_{j=2}^{n} \left( (\Omega_{0,j} \cup C_j^*) \cup (\Omega^*_0,j \cup C_j^*) \right) \right).$$

Repeating this process, we can construct a planar Riemann surface $\Omega_n$ from $\Omega_{n-1}$ by attaching copies of $\hat{D}$ along all boundary components of $\Omega_{n-1}$ for every $n \in \mathbb{N}$. Thus we obtain a planar uniformization, i.e, a planar cover,
\[ \tilde{R} = \bigcup_{n=0}^{\infty} \Omega_n \] of a closed Riemann surface \( R \) of genus \( n - 1 \) obtained from \( \Omega_0 \) by identifying every \( C_j \) with \( C_j^* \) under the action of \( g_j \). Here, recall that \( g_j \) and \( h_j \), and hence every \( g_j \) and \( h_j \), which can be extended canonically to a conformal homeomorphism of \( \tilde{R} \) onto itself, satisfy that \( g_j = h_j^{-1} \) for every \( j \).

Now, by the uniformization theorem due to Klein, Poicaré, and Koebe, we can consider \( \tilde{R} \) as a planar domain, which is denoted by \( \Omega \). Since the complement of \( \Omega \) belongs to \( N_D \), every \( g_j : \Omega \to \Omega \) is a Möbius transformation, and we may assume that \( D \), considered as a sub-domain of \( \Omega \), has the boundary \( \{ \tilde{C}_j, \tilde{C}_j^* \mid j = 2, \ldots, n \} \) such that the exterior of \( \tilde{C}_j \) is mapped by the Möbius transformation \( g_j \) onto the interior of \( C_j^* \) for every \( j \). Hence the group \( G \) generated by all \( g_j \) is a Schottky group with respect to \( \tilde{C} = \{ \tilde{C}_j, \tilde{C}_j^* \mid j = 2, \ldots, n \} \), or equivalently, that \( \tilde{R} \) is a Schottky uniformization of \( R \) with respect to \( \tilde{C} \).

Finally, the anti-conformal involution \( \tau \) of \( \tilde{D} \), which can be extended to an anti-conformal involution of \( \tilde{R} \), should be a Möbius transformation precomposed by the complex conjugate. Since \( \tau \) has \( \tilde{C}_1 \) as the set of fixed points, \( \tilde{C}_1 \) should be a circle, and hence so is all curves \( \tilde{C}_j \) and \( \tilde{C}_j^* \), which implies the first assertion.

The second assertion is clear from Proposition 3.

\[ \square \]

References


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Consider a set
\[ \mathcal{C} = \{ C_j, C'_j \mid j \in \mathbb{N} \} \]
of countably infinite number of pairs of simple closed curves in \( \mathbb{C} \) such that not only these curves but also the interiors of them are mutually disjoint. We further assume that the exterior of \( C_j \) is mapped onto the interior of \( C'_j \) by a Möbius transformation \( g_j \) for every \( j \). Let \( G \) be the group generated by all \( g_j \) defined as above. Then we first show that, if \( \mathcal{C} \) satisfies the modified Maskit condition and the tameness condition, \( G \) is an infinitely generated Schottky group with respect to \( \mathcal{C} \).

Here, we call \( G \) an infinitely generated Schottky group with respect to the loop family \( \mathcal{C} \), if the limit set \( \Lambda(G) \) of \( G \) is totally disconnected. If all elements of \( \mathcal{C} \) are circles, then we call \( G \) an infinitely generated classical Schottky group.

Finally, as the main result of this paper, we show that, letting \( G \) be an infinitely generated Schottky group satisfying the modified Maskit condition and the tameness condition, if the corresponding Schottky marked Riemann surface \( R \) is maximally symmetric, \( G \) is an infinitely generated classical Schottky group.