Estimations of distances for Heegaard splitting, and for bridge decomposition

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Contents

1 Backgrounds and basic definitions 3
  1.1 Introduction ................................................. 3
  1.2 Preliminaries ............................................. 9
     1.2.1 Heegaard splittings .................................. 9
     1.2.2 Bridge decompositions .............................. 10
  1.3 Curve complexes ........................................ 13

2 Estimations for Heegaard splitting, and for bridge decom¬
position 15
  2.1 Rubinstein-Scharlemann graphics .......................... 15
     2.1.1 Sweep-outs induced from Heegaard splittings ...... 15
     2.1.2 Labelling regions of the graphic induced from Heegaard
            splittings ........................................... 18
     2.1.3 Sweep-outs induced from \((g, b)\)-bridge decompositions .. 21
  2.2 An estimation of Hempel distance by using Reeb Graph .... 23
  2.3 Proof of Theorem 1.1.1 .................................... 26
  2.4 Proofs of Theorem 1.1.2 and Corollary 1.1.3 .......... 34

3 Heegaard splitting with distance exactly \(n\) 37
  3.1 A pair of curves with distance exactly \(n\) .................. 37
     3.1.1 Subsurface projection maps .......................... 37
     3.1.2 A pair of curves with distance exactly \(n\) .......... 38
  3.2 Heegaard splitting with distance exactly \(n\) for each integer \(n > 0\) 42
     3.2.1 Disk complex for a special compression body ....... 42
     3.2.2 Heegaard splitting with distance exactly \(n\) .......... 44
     3.2.3 \((1, 1)\)-splitting with distance exactly \(n\) .......... 45
Chapter 1

Backgrounds and basic definitions

1.1 Introduction

In 1970's, Thurston revealed that hyperbolic geometry is essential for the research of 3-dimensional manifold (: 3-manifold). Since then the research field has developed the area until now.

In 1968, Waldhausen[33] made a breakthrough in the research of 3-manifolds, that is, he showed that a naive algebraic structure of a 3-manifold, called the fundamental group, determines the topology of the manifold provided it is irreducible and sufficiently large, where a 3-manifold is called sufficiently large if it contains an incompressible surface. (Here we note that every compact 3-manifold is uniquely decomposed into irreducible 3-manifolds with very few exceptions, hence, the irreducibility is not essential.) Since then, incompressible surfaces together with the technique introduced by Waldhausen, called hierarchy of the manifold, have been playing essential role in the theory of 3-manifolds.

For a compact orientable 3-manifold $M$, a Heegaard splitting of $M$ is a decomposition $M = C_1 \cup_P C_2$, where $P = \partial_+ C_1 = \partial_+ C_2$ is a closed embedded separating surface in $M$ and $C_i (i = 1, 2)$ is a compression body. The surface $P$ is called a Heegaard surface, and the genus of $P$ is the genus of this Heegaard

\[1\] A surface in a 3-manifold is called incompressible if it does not admit a disk which compresses an essential simple closed curve in it.
splitting (for details, see subsection 1.2.1). This concept was introduced by Heegaard[10] in 1898. Moreover, Moise[26] and Bing[3] showed that every compact orientable 3-manifold admits a Heegaard splitting. It is known that some properties of ambient 3-manifolds are reflected to Heegaard splittings of the 3-manifolds. For example, Haken[7] showed that a Heegaard splitting of a reducible 3-manifold is reducible.

In 1987, Casson and Gordon[4] introduced a new concept in Heegaard theory, called strong irreducibility, and showed that if a Heegaard splitting is not strongly irreducible (this property is called weakly reducible), then either the Heegaard splitting is reduced to a low genus Heegaard splitting, or the ambient 3-manifold contains an incompressible surface. Afterwards, the concept turned out to be very useful. For example, each strongly irreducible Heegaard splitting behave very nicely with respect to incompressible surfaces, or another strongly irreducible Heegaard splitting, and this fact furnishes a powerful method for analyzing given Heegaard splittings (see, for example, [29]).

**Distance of Heegaard splitting**

In 2001, Hempel[11] introduced the concept of distance of Heegaard splitting, and showed that a Heegaard splitting \( C_1 \cup_P C_2 \) is reducible (resp. weakly reducible) if and only if the distance of the Heegaard splitting, denoted by \( d(P) \), is 0 (resp. \( \leq 1 \)), (hence, we may say that the distance gives a vast refinement of the concept of strong irreducibility) and it is known that if a 3-manifold admits a Heegaard splitting with distance at least 3, then the manifold is hyperbolic. In succession, several authors showed that the concept well represents various complexities of 3-manifolds. For example, Hartshorn[9] showed that the Euler characteristic of an incompressible surface in a 3-manifold bounds the distances of the Heegaard splittings of the manifold, and Scharlemann and Tomova[31] showed that if a 3-manifold admits a high distance Heegaard splitting, then certain "rigidly" holds for Heegaard splittings of the 3-manifold. More precisely:

**Theorem 1.**[31, Corollary 4.5] _If a compact orientable 3-manifold \( M \) has a genus-\( g \) Heegaard surface \( P \) with distance \( d(P) > 2g \), then the following holds._

1. \( P \) is a minimal genus Heegaard surface of \( M \).
2. any other Heegaard surface of the same genus is isotopic to \( P \).
3. moreover, any Heegaard surface \( Q \) of \( M \) with \( 2g(Q) \leq d(P) \) is isotopic to a stabilization or boundary stabilization of \( P \).

Roughly speaking, the above theorem shows that if the distance of \( P \) is greater than \( 2g(P) \), then \( P \) is "rigid". Moreover, Berge and Scharlemann[2] made a detailed analysis for the case \( g(P) = 2 \), and showed the following fact:

**Fact 1.** If a closed orientable 3-manifold \( M \) has a genus 2 Heegaard surface \( P \) with distance \( d(P) = 4 \), that is, \( d(P) = 2g(P) \) holds, then any other genus-2 Heegaard surfaces of \( M \) is isotopic to \( P \). Moreover there exist examples of 3-manifolds each of which admits mutually non-isotopic genus 2 Heegaard surfaces with distance 3. (Note that, in the case \( d(P) = 2g(P) - 1 \) holds.)

Hence it seems that it is natural to call \( P \) with distance \( d(P) = 2g(P) \) a critical distance Heegaard surface. Therefore we pose the following question:

**Question 1.** If a compact orientable 3-manifold \( M \) has a genus \( g \) Heegaard surface \( P \) with distance \( d(P) = 2g(P) \), then are any other genus-\( g(P) \) Heegaard surfaces of \( M \) isotopic to \( P \)?

In Section 2.3, we tackle the above question. Unfortunately, we do not have the answer for Question 1 yet. However toward this ends, we could show that any other genus-\( g(= g(P)) \) Heegaard surface \( Q \) with \( d(Q) = 2g \) can be isotoped to a neat position with respect to a height function induced from the genus-\( g \) Heegaard surface \( P \). Precisely speaking:

**Theorem 1.1.1.** Suppose that a closed orientable 3-manifold \( M \) admits a genus-\( g \) Heegaard surface \( P \) with distance \( d(P) = 2g \). Let \( Q \) be another genus-\( g \) Heegaard surface which is strongly irreducible. Then there is a height function \( f : M \rightarrow I \) induced from \( P \) (that is, the level surfaces of \( f \) with height \( t(\neq 0,1) \) are isotopic to \( P \)) such that by isotopy, \( Q \) is deformed into a position satisfying the following:

1. \( f|_Q \) has \( 2g + 2 \) critical points \( p_0 < p_1 < \cdots < p_{2g+1} \) where \( p_0 \) is a minimum and \( p_{2g+1} \) is a maximum, and \( p_1, \ldots, p_{2g} \) are saddles,

2. if we take regular values \( r_i \) \( (i = 1, \ldots, 2g + 1) \) such that \( f(p_{i-1}) < r_i < f(p_i) \), then \( f^{-1}(r_i) \cap Q \) consists of a circle if \( i \) is odd, and \( f^{-1}(r_i) \cap Q \) consists of two circles if \( i \) is even.
The main tool of the proof of Theorem 1.1.1 is *Reeb graphs* derived from horizontal arcs of the *Rubinstein-Scharlemann graphic* (or *graphic* for short). We give a short explanation about these ideas. Graphic was introduced by Rubinstein and Scharlemann[29] for studying Reidemeister-Singer distance of two strongly irreducible Heegaard splittings. Moreover, under certain technical conditions, Li[21] showed that there exist horizontal arcs in graphics disjoint from the union of the regions labelled $X, x, Y$ or $y$ (for the definitions, see subsection 2.1.2), and gave an alternative proof of Theorem 1 by using such horizontal arcs. On the other hand, Johnson[15] introduced Reeb complex of graphic to improve the estimation of Reidemeister-Singer distance given by Rubinstein and Scharlemann[29]. In [13], we combined the ideas (horizontal arc, Reeb complex) to define *Reeb graphs*, and to give an estimation of Hempel distance (see Theorem 2.2.1).

**Distance of bridge decomposition**

The concept of distance of Heegaard splitting have been extended to *bridge surfaces* for links in closed 3-manifolds, and have been studied by several authors. For example, Bachman and Schleimer[2] proved that Hartshorn's results[9] can be extended to the distances of bridge surfaces, and Tomova[32] proved that Scharlemann and Tomova's results[31] can be extended to the distances of bridge surfaces. Recently, Johnson and Tomova[17] proved that Tomova's results[32] can be extended to bridge surfaces for tangles in compact 3-manifolds, and Jang[14] showed that a result of Bachman and Schleimer[2] can be improved in the case when the essential surfaces under consideration are 2-spheres.

In Section 2.4, we show that a result of Tomova[32, Theorem 10.3] can be improved in the case of two different bridge spheres for a link.

**Theorem 1.1.2.** Suppose $L$ is a link in $S^3$ and $P$ is a bridge sphere for $L$ with $|P \cap L| \geq 6$. If $Q$ is another bridge sphere for $L$ such that $Q$ is not equivalent to $P$, then $d(P, L) \leq |Q \cap L| - 2$, where $d(P, L)$ denotes the distance of the bridge sphere $P$ and $|\cdot|$ denote the number of connected components.

By the above theorem, we have the next corollary, which improves [32, Corollary 10.7] in the case of bridge sphere.
Corollary 1.1.3. If $P$ is a bridge sphere for a link $L$ such that $d(P, L) > |P \cap L| - 2$, then the minimal bridge sphere for $L$ is unique up to isotopy rel $P \cap L$.

Construction of Heegaard splitting with distance exactly $n$

Hempel[11] showed that there exist arbitrarily high distance Heegaard splittings of closed 3-manifolds by using a construction of Kobayashi[20]. Moreover, Evans[5] improved the result of Hempel[11] by constructing a Heegaard splitting of genus $g$ with distance at least $n$ for given $g > 1$ and $n \geq 0$.

Here it will be natural to ask the following.

Question 2. Given $g > 1$ and $n \geq 0$, does there exist a genus-$g$ Heegaard splitting with distance exactly $n$?

For certain values, there are known examples that answer the above question affirmatively. For example, recall that Berge-Scharlemann[2] showed that there exist 3-manifolds each of which admits genus-2 Heegaard splittings with distance exactly 3 (see Fact 2).

In Section 3.2, we first construct Heegaard splittings of compact orientable 3-manifolds with nonempty boundaries which have distance exactly $n$.

Theorem 1.1.4. For any integer $n > 0$ and any integer $g > 1$, there exists a genus-$g$ Heegaard splitting $C_1 \cup_P C_2$ with distance exactly $n$, where $C_1$ and $C_2$ are genus-$g$ compression bodies.

Here is a sketch of the proof of Theorem 1.1.4. In [28], Schleimer gave a method of constructing a pair of curves on the five-holed sphere with distance exactly four by using subsurface projection maps defined by Masur and Minsky[24] (for the definition, see Subsection 3.1.1). In Subsection 3.1.2, we mimic the idea of Schleimer[28] to construct a pair of curves with distance exactly $n$ for any positive integer $n$. By using the pair of curves and the properties of a compression body obtained by adding a 1-handle to $S \times [0, 1]$ where $S$ is a closed surface (for detail, see Subsection 3.2.1), we give the proof of Theorem 1.1.4.

Moreover, we apply the above idea to $(1,1)$-splittings, and we show that;

Theorem 1.1.5. For any even integer $n > 0$, there exists a $(1,1)$-splitting with distance exactly $n$. 

7
Theorems 1.1.4 and 1.1.5 are joint work with Yeonhee Jang and Tsuyoshi Kobayashi.
1.2 Preliminaries

Throughout this paper, we work in the differential category. Let $S$ be a compact orientable surface with genus $g$ and $p$ boundary components. A simple closed curve in $S$ is inessential if it bounds a disk in $S$ or is parallel to $\partial S$. A simple closed curve in $S$ is essential if it is not inessential. An arc properly embedded in $S$ is inessential if it co-bounds a disk in $S$ together with an arc on $\partial S$. An arc properly embedded in $S$ is essential if it is not inessential. We say that $S$ is sporadic if $g = 0, p \leq 4$ or $g = 1, p \leq 1$. We say that $S$ is simple, if $S$ contains no essential simple closed curves. We note that $S$ is simple if and only if $S$ is a 2-sphere with at most three boundary components. A subsurface $X$ in $S$ is essential if each component of $\partial X$ is contained in $\partial S$ or is essential in $S$.

1.2.1 Heegaard splittings

A 3-manifold $V$ is a compression body if there exists a closed (possibly empty) surface $F$ and a 0-handle $B$ such that $V$ is obtained from $F \times [0,1] \cup B$ by adding 1-handles to $F \times \{1\} \cup \partial B$. The subsurface of $\partial V$ corresponding to $F \times \{0\}$ is denoted by $\partial_0 V$. Then $\partial_+ V$ denotes the subsurface $\text{cl}(\partial V - \partial_0 V)$ of $\partial V$, where $\text{cl}(\cdot)$ denotes the closure. The genus of $\partial_+ V$ is called the genus of $V$. A compression body $V$ is called a handlebody if $\partial_0 V = \emptyset$. A disk $D$ properly embedded in $V$ is called an essential disk if $\partial D$ is an essential simple closed curve in $\partial_+ V$. A spine $\Sigma$ of $V$ is a properly imbedded 1-complex which is strong deformation retract of $V$. By a technical reason, throughout this paper, we suppose that each vertex of spines of genus-$g (> 1)$ compression bodies has valency three (for a detailed discussion, see Section 2 of [19]).

![Figure 1.1: a spine of a genus-3 handlebody](image)

Let $M$ be a compact orientable 3-manifold. We say that $C_1 \cup_P C_2$ is a
genus-\(g\) Heegaard splitting of \(M\) if \(C_1\) and \(C_2\) are genus-\(g\) compression bodies in \(M\) such that \(C_1 \cup C_2 = M\) and \(C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P\). Then \(P\) is called a (genus-\(g\)) Heegaard surface of \(M\).

If there are essential disks \(D_1, D_2\) in \(C_1, C_2\) respectively so that \(\partial D_1 = \partial D_2\), \(C_1 \cup_P C_2\) is said to be reducible. If there are essential disks \(D_1, D_2\) in \(C_1, C_2\) respectively so that \(\partial D_1, \partial D_2\) are disjoint on \(P\), \(C_1 \cup_P C_2\) is said to be weakly reducible. It is easy to see that if \(C_1 \cup_P C_2\) is reducible, it is weakly reducible. If \(C_1 \cup_P C_2\) is not weakly reducible, it is said to be strongly irreducible. A Heegaard splitting \(M = C_1 \cup_P C_2\) is stabilized if there are essential disks \(D_1, D_2\) in \(C_1, C_2\) respectively such that \(\partial D_1\) and \(\partial D_2\) intersects transversely in a single point. We note that a genus-\(g\) Heegaard splitting \(C_1 \cup_P C_2\) is stabilized if and only if there exists a genus-\((g - 1)\) Heegaard splitting \(C'_1 \cup_P C'_2\) such that \(C_1 \cup_P C_2\) is obtained from \(C'_1 \cup_P C'_2\) by adding a "trivial" handle. We say that \(C''_1 \cup_P C''_2\) is a stabilization of \(C_1 \cup_P C_2\), if \(C''_1 \cup_P C''_2\) is obtained from \(C_1 \cup_P C_2\) by adding trivial handles.

1.2.2 Bridge decompositions

Let \(M\) be a compact orientable 3-manifold, \(\gamma\) a union of mutually disjoint arcs or simple closed curves properly embedded in \(M\), and \(F\) a surface embedded in \(M\), which is in general position with respect to \(\gamma\). A surface \(D\) in \(M\) is a \(\gamma\)-disk, if \(D\) is a disk intersecting \(\gamma\) in at most one transverse point. Let \(\ell(\subset F)\) be a simple closed curve with \(\ell \cap \gamma = \emptyset\). We say that \(\ell\) is \(\gamma\)-inessential if \(\ell\) bounds a \(\gamma\)-disk in \(F\), and \(\ell\) is \(\gamma\)-essential if it is not \(\gamma\)-inessential. We say that a surface \(D\) is a \(\gamma\)-compressing disk for \(F\) if; \(D\) is a \(\gamma\)-disk, \(D \cap F = \partial D\), and \(\partial D\) is a \(\gamma\)-essential simple closed curve in \(F\). Let \(F_1, F_2\) be surfaces in \(M\) which are in general position with respect to \(\gamma\). We say that \(F_1\) and \(F_2\) are \(\gamma\)-parallel if they co-bound a 3-manifold homeomorphic to \(F_1 \times [0, 1]\) intersecting \(\gamma\) in vertical arcs, where \(F_1 = F_1 \times \{0\}\) and \(F_2 = F_1 \times \{1\}\). We say that \(F_1\) and \(F_2\) are \(\gamma\)-isotopic if there exists an isotopy from \(F_1\) to \(F_2\) so that \(F_1\) remains transverse to \(\gamma\) throughout the isotopy.
Handlebodies containing trivial arcs

Let $H$ be a genus-$g (\geq 0)$ handlebody. Recall that if $g > 1$, spine $\Sigma_H$ of $H$ is a 1-complex contained in the interior of $H$, which is a strong deformation retract of $H$, where each vertex of $\Sigma_H$ has valency three. For a genus-0 handlebody (the 3-ball), we let the spine be a point in the interior of the 3-ball, and for a genus-1 handlebody (solid torus), we let the spine be a core circle of the solid torus. We note that $H \setminus \Sigma$ is homeomorphic to $\partial H \times [0, 1)$, and we fix this identification. We say that a set of $n$-arcs $\{t_1, \ldots, t_n\}$ properly embedded in $H$ is a set of trivial $n$-arcs if $t_1 \cup \cdots \cup t_n$ is parallel to $\partial H$. Let $H$ be a handlebody and $\tau = \{t_1, \ldots, t_n\}$ a set of trivial $n$-arcs in $H$. Then $\tau$ can be isotoped in $H$ so that the projection from $\partial H \times [0, 1)$ to $[0, 1)$ has exactly one critical point in each $t_i$. For the pair $(H, \tau)$, we let the spine $\Sigma_{(H, \tau)}$ be the union of $\Sigma_H$ together with a collection of straight arcs $\alpha_1, \ldots, \alpha_n$, where one endpoint of each $\alpha_i$ lies on the critical point of $t_i$, and the other endpoint lies on $\Sigma_H$.

![Figure 1.2: a spine of a genus-2 handlebody containing a set of trivial 3-arcs](image)

$(g, n)$-bridge decompositions

Let $M$ be a closed orientable 3-manifold, and $L$ a link in $M$. We say that $(A, \tau_A) \cup P (B, \tau_B)$ is a $(g, n)$-bridge decomposition ($n > 0$) for the pair $(M, L)$ if $P$ separates $(M, L)$ into two components $(A, \tau_A), (B, \tau_B)$ where $\tau_A = L \cap A$ (resp. $\tau_B = L \cap B$) is a set of trivial $n$-arcs in $A$ (resp. $B$). Then we say that $P$ is a $(g, n)$-bridge surface, or a bridge surface for short. It is known that each $(M, L)$ has a $(g, n)$-bridge decomposition for some $g$ and $n$ (for a detailed discussion, see [12, Lemma 2.1]).

Given a $(g, n)$-bridge decomposition $(A, \tau_A) \cup P (B, \tau_B)$ for $(M, L)$, there are three ways to create new bridge surfaces for $(M, L)$: (1) adding trivial one-
handles disjoint from $L$ to $P$ \textit{(stabilizing)}, (2) adding dual one-handles where one of them has an arc of $L$ as its core \textit{(meridionally stabilizing)}, and (3) introducing a pair of a canceling minimum and maximum for $L$ \textit{(perturbing)} (for details, see [32, Figure 15]). We say that another bridge surface $Q$ is \textit{equivalent} to $P$ if $Q$ is $L$-isotopic to a copy of $P$ which may have been stabilized, meridionally stabilized and perturbed.
1.3 Curve complexes

Let $S$ be a compact orientable surface. Except in sporadic cases, the curve complex $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on $S$, and a collection of $k+1$ vertices forms a $k$-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in $S$. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. Note that a sporadic surface $S$ is simple unless $S$ is a torus, a torus with one boundary component, or a sphere with 4 boundary components. When $S$ is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k+1$ vertices forms a $k$-simplex of $\mathcal{C}(S)$ if they can be realized by curves in $S$ which mutually intersect exactly once (resp. twice). The arc-and-curve complex $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on $S$, and a collection of $k+1$ vertices forms a $k$-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in $S$.

Throughout this paper, for a vertex $x \in \mathcal{C}(S)$ we often abuse notations to use $x$ to represent (the isotopy class of) a geometric representative of $x$.

For two vertices $x, y$ of $\mathcal{C}(S)$, we define the distance $d_{\mathcal{C}(S)}(x, y)$ between $x$ and $y$, which will be denoted by $d_S(x, y)$ in brief, as the minimal number of 1-simplexes of a simplicial path in $\mathcal{C}(S)$ joining $x$ and $y$. Let $X, Y$ be subsets of the vertices of $\mathcal{C}(S)$. Then we define $d(X, Y) := \min\{d(x, y) \mid x \in X, y \in Y\}$ and $\text{diam}_S(X, Y) :=$ the diameter of $X \cup Y$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x, y)$, $d_{\mathcal{AC}(S)}(X, Y)$ and $\text{diam}_{\mathcal{AC}(S)}(X, Y)$. For a sequence $a_0, a_1, \ldots, a_n$ of vertices in $\mathcal{C}(S)$ with $a_i \cap a_{i+1} = \emptyset$ ($i = 0, 1, \ldots, n-1$), we denote by $[a_0, a_1 \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1 \ldots, a_n$ in this order. We say that a path $[a_0, a_1 \ldots, a_n]$ is a geodesic if $n = d_S(a_0, a_n)$.

Let $V$ be a compression body containing a set of trivial arcs $\tau$. Then the disk complex $\mathcal{D}(V \setminus \tau)$ is the subcomplex of $\mathcal{C}(\partial_+ V \setminus \partial \tau)$ consisting of the vertices with representatives bounding disks of $V \setminus \tau$. Note that if $\tau$ is empty, $\mathcal{D}(V)$ is the subcomplex of $\mathcal{C}(\partial_+ V)$ consisting of the vertices with representatives bounding disks of $V$.

For a genus-$g$ ($g \geq 2$) Heegaard splitting $C_1 \cup_S C_2$ (or a Heegaard surface $S$),
the (Hempel) distance of $C_1 \cup S C_2$ (or $S$) is defined by $d(S) := d_S(D(C_1), D(C_2))$. Similarly we can define the distance of a bridge decomposition as follows. Suppose that $M$ is a closed orientable 3-manifold containing a link $L$, and $(A, \tau_A) \cup_P (B, \tau_B)$ is a bridge decomposition for $(M, L)$ (or $P$ is a bridge surface for $(M, L)$). Then the distance of $(A, \tau_A) \cup_P (B, \tau_B)$ (or bridge surface $P$) is defined by $d(P, L) := d(D(A \setminus \tau_A), D(B \setminus \tau_B))$. 
Chapter 2

Estimations for Heegaard splitting, and for bridge decomposition

2.1 Rubinstein-Scharlemann graphics

2.1.1 Sweep-outs induced from Heegaard splittings

Let $M$ be a smooth closed orientable 3-manifold. A sweep-out is a smooth map $f : M \to I$ such that for each $x \in (0, 1)$, the level set $f^{-1}(x)$ is a closed surface, and $f^{-1}(0)$ (resp. $f^{-1}(1)$) is a connected, finite graph such that each vertex has valency three. Each of $f^{-1}(0)$ and $f^{-1}(1)$ is called a spine of the sweep-out. It is easy to see that each level surface of $f$ is a Heegaard surface of $M$ and the spines of the sweep-outs are spines of the two handlebodies in the Heegaard splitting. Conversely, given a Heegaard splitting $A \cup B$ of $M$, it is easy to see that there is a sweep-out $f$ of $M$ such that each level surface of $f$ is isotopic to $P$, $f^{-1}(0)$ is a spine of $A$, and $f^{-1}(1)$ is a spine of $B$. We call it a sweep-out obtained from $A \cup B$.

Given two sweep-outs, $f$ and $g$ of $M$, we consider their product $f \times g$ (that is, $(f \times g)(x) = (f(x), g(x))$), which is a smooth map from $M$ to $I \times I$. Kobayashi and Saeki [19] has shown that by arbitrarily small deformations of $f$ and $g$, we can suppose that $f \times g$ is a stable map on the complement of the four spines. At each point in the complement of the spines, the differential of the
map $f \times g$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$. This map have a one dimensional kernel for a generic point in $M$. The discriminant set for $f \times g$ is the set of points where the differential has a higher dimensional kernel. Mather's classification of stable maps [18] implies that at each point of the discriminant set, the dimension of the kernel of the differential is two, and the discriminant set is a one dimensional smooth submanifold in the complement of the spines in $M$. Moreover the discriminant set consists of all such $x$ that a level surface of $f$ is tangent to a level surface of $g$ at $x$ (here, we note that the tangent point is either a “center” or “saddle”).

Let $f$, $g$ be as above with $f \times g$ stable. The image of the discriminant set is a graph in $I \times I$, which is called the Rubinstein-Scharlemann graphic. It is known that the Rubinstein-Scharlemann graphic is a finite 1-complex $\Gamma$ with each vertex having valency four or two. Each valency four vertex is called a crossing vertex, and each valency two vertex is called a birth-death vertex. There are valency one or two vertices of the graphic on the boundary of $I \times I$. Each component of the complement of $\Gamma$ in $I \times I$ is called a region. At each point of a region, the corresponding level surfaces of $f$ and $g$ are disjoint or

![Figure 2.1: center](image)

![Figure 2.2: saddle](image)
intersect transversely.

Figure 2.3: a graphic of genus-2 Heegaard splittings for $S^3$

Figure 2.4: intersections of Heegaard surfaces corresponding to points of a graphic

The stable map $f \times g$ is \textit{generic} if each arc $\{s\} \times I$ or $I \times \{t\}$ contains at most one vertex of the graphic. By Proposition 6.14 of [19], by arbitrarily small deformation of $f$ and $g$, we may suppose that $f \times g$ is generic.
2.1.2 Labelling regions of the graphic induced from Heegaard splittings

Let \( f \) and \( g \) be sweep-outs obtained from Heegaard splittings \( A \cup P B, X \cup Q Y \), respectively with \( f \times g \) stable. Now we introduce how to label each region of the graphic with following the convention of [29]. For each \( s \in I \), we put that \( P_s = f^{-1}(s) \) (in particular, \( \Sigma_A \) denotes \( P_0 \) and \( \Sigma_B \) denotes \( P_1 \)), \( A_s = f^{-1}([0, s]) \) and \( B_s = f^{-1}([s, 1]) \). Similarly, for \( t \in I \), we put that \( Q_t = g^{-1}(t) \) (in particular, \( \Sigma_X \) denotes \( Q_0 \) and \( \Sigma_Y \) denotes \( Q_1 \)), \( X_t = g^{-1}([0, t]) \) and \( Y_t = g^{-1}([t, 1]) \). Let \((s, t)\) be a point in a region of the graphic. Then either \( P_s \cap Q_t = \emptyset \), or \( P_s \) and \( Q_t \) intersect transversely in a collection \( C = \{c_1, \ldots, c_n\} \) of simple closed curves.

**Definition 2.1.1.** Let \( C = \{c_1, \ldots, c_n\} \) be as above. Then \( C_P \) denotes the subset of \( C \) consisting of the elements which are essential on \( P_s \). Furthermore the subset \( C_A \) of \( C_P \) is defined by:

\[
C_A = \{c \in C_P \mid c \text{ bounds a disk } D \text{ in } Q_t \setminus C_P \text{ such that } N(\partial D, D) \subset A_s \},
\]

where \( N(\partial D, D) \) is a regular neighborhood of \( \partial D \) in \( D \). Analogously \( C_B \subset C_P \) and \( C_X, C_Y \subset C_Q \) are defined.

\[\text{Figure 2.5: intersections of } P_s \text{ and } Q_t\]
Lemma 2.1.2. [29, Lemma 4.5] Suppose that $C_P$ and $C_Q$ are empty, and there exists a meridian disk $D$ in $A_s$ which intersects $Q_t$ only in inessential simple closed curves. Moreover, suppose that there is an essential simple closed curve $l$ on $Q_t$ such that $l \subset A_s$. Then either $A \cup_P B$ is weakly reducible or $M$ is the 3-sphere $S^3$. The statement obtained by substituting $(A, P, Q)$ in the above with $(B, P, Q), (X, Q, P)$ or $(Y, Q, P)$ also hold.

If $C_A$ (resp. $C_B, C_X, C_Y$) is non-empty, the region is labelled $A$ (resp. $B, X, Y$). If $C_P$ and $C_Q$ are both empty and $A_s$ (resp. $B_s$) contains an essential curve of $Q_t$, then the region is labelled $b$ (resp. $a$). If $C_P$ and $C_Q$ are both empty and $X_t$ (resp. $Y_t$) contains an essential curve of $P_s$, then the region is labelled $y$ (resp. $x$). $R_A$ (resp. $R_B, R_X, R_Y, R_a, R_b, R_x, R_y$) denotes the closure of the union of the regions labelled $A$ (resp. $B, X, Y, a, b, x, y$). $R_t$ denotes the closure of the union of the unlabelled regions.

Lemma 2.1.3. [29, Corollary 5.1] If there exist two adjacent regions such that one is labelled $A$ (resp. $X$) and the other is labelled $B$ (resp. $Y$), then $A \cup_P B$ (resp. $X \cup_Q Y$) is weakly reducible.

The proof of the next lemma can be found in the paragraph preceding [29, Proposition 5.9].

Lemma 2.1.4. Suppose that $A \cup_P B$ and $X \cup_Q Y$ are strongly irreducible and $M \neq S^3$. Then each region adjacent to $\{0\} \times I$ (resp. $\{1\} \times I, I \times \{0\}, I \times \{1\}$) is labelled $A$ or $a$ (resp. $B$ or $b$, $X$ or $x$, $Y$ or $y$).

The next lemma is proved in the proof of Lemma 3.2 of [21](see also [16], [13]).

Lemma 2.1.5. Let $M, P, Q$ be as above. Suppose $P, Q$ are strongly irreducible. If $Q$ is not isotopic to $P$, then there exists $t \in (0, 1)$ such that $(0, 1) \times \{t\}$ is disjoint from $R_X \cup R_s$ and $R_Y \cup R_y$.

Proof. Suppose that there exists $t$ such that for values $s_-, s_+ \in (0, 1), (s_\pm, t) \in R_X \cup R_s$ and $(s_\mp, t) \in R_Y \cup R_y$. We have the following cases.

Case 1. $(s_\pm, t) \in R_X, (s_\mp, t) \in R_Y$.

Without loss of generality, we may suppose that $(s_-, t) \in R_X$ and $(s_+, t) \in R_Y$. In this case, $P_{s_-} \cap Q_t$ contains a simple closed curve which is essential.
on $Q_t$ and bounds a disk in $X_t$ while $P_{s+} \cap Q_t$ contains a simple closed curve which is essential on $Q_t$ and bounds a disk in $Y_t$. This shows that $Q$ is weakly reducible, a contradiction.

**Case 2.** $(s_{\pm}, t) \in R_x, (s_{\mp}, t) \in R_y$.

Without loss of generality, we may suppose that $(s_{-}, t) \in R_x$ and $(s_{+}, t) \in R_y$. In this case, by an isotopy, we may suppose that $Q_t$ is contained in $f^{-1}([s_{-}, s_{+}]) \cong P \times [s_{-}, s_{+}]$. If $Q_t$ is incompressible in $P \times [s_{-}, s_{+}]$, $Q$ is isotopic to $P$ (Corollary 3.2 of [33]), a contradiction. If $Q_t$ is compressible in $P \times [s_{-}, s_{+}]$ then there is a compression disk $D$ such that $D \subset P \times [s_{-}, s_{+}]$. $D$ is contained in $X_t$ or $Y_t$. By applying Lemma 2.1.2 to $P_{s_{-}}, Q_t$ and $D$ (if $P_{s_{-}}$ and $\text{int}D$ are contained in the same component of $M \setminus Q_t$) or $P_{s_{+}}, Q_t$ and $D$ (if $P_{s_{+}}$ and $\text{int}D$ are contained in the same component of $M \setminus Q_t$), we see that $Q$ is weakly reducible, a contradiction.

**Case 3.** $(s_{\pm}, t) \in R_x, (s_{\mp}, t) \in R_Y$ (or $(s_{\pm}, t) \in R_X, (s_{\mp}, t) \in R_y$).

Without loss of generality, we may suppose that $(s_{-}, t) \in R_x$ and $(s_{+}, t) \in R_y$. In this case, since $(s_{-}, t) \in R_x$, each component of $P_{s_{-}} \cap Q_t$ is inessential on both $P_{s_{-}}$ and $Q_t$, and there is an essential simple closed curve $\ell$ in $P_{s_{-}}$ such that $\ell \subset Y_t$. Let $C^+$ be the collection of simple closed curve(s) consisting of $P_{s_{+}} \cap Q_t$, then $C^+_P$ denotes the subset of $C^+$ which are essential on $Q_t$. Since $(s_{+}, t) \in R_y$, there is a disk component, say $E$, of $P_{s_{+}} \setminus Q_t$ such that $N(\partial E, E) \subset Y_t$. Since $M$ admits a strongly irreducible Heegaard splitting, $M$ is irreducible. Hence there is an ambient isotopy $\psi_t$ ($0 \leq t \leq 1$) of $M$ realizing disk swaps between $E$ and $Q_t$ such that $\psi_1(E)$ is a meridian disk of $Y_t$. Here we note that each component of $\psi_1(P_{s_{-}}) \cap Q_t$ is inessential on both $\psi_1(P_{s_{-}})$ and $Q_t$, and there is an essential simple closed curve $\ell'$ in $\psi_1(P_{s_{-}})$ such that $\ell' \subset Y_t$. By applying Lemma 2.1.2 to $Q_t$, $\psi_1(P_{s_{-}})$, and $\psi_1(E)$, we see that $Q$ is weakly reducible, a contradiction.

We note that the arguments in the proof of [16, Lemma 21] work for the arc in Lemma 2.1.5. Hence we have:

**Lemma 2.1.6.** If $g$ strongly splits $f$, there exists $t$ such that $I \times \{t\}$ is disjoint from $(R_X \cup R_x) \cup (R_Y \cup R_y)$ and the restriction of $f$ to $Q_t$ is a Morse function such that for each regular value $s$, $P_s \cap Q_t$ contains a simple closed curve which is essential on $P_s$.  

20
Lemma 2.1.7. Let $t$ be as in Lemma 2.1.5. There is a subarc $[s_0, s_1] \times \{t\} \subset I \times \{t\}$ such that:

- $(s_0, t) \in \{an \text{ edge of the graphic}\}$,
- $(s_1, t) \in \{an \text{ edge of the graphic}\}$, and
- for any $s \in (s_0, s_1)$, $(s, t) \in R_\phi$, and for any small $\epsilon > 0$, $(s_0 - \epsilon, t) \in R_A$ and $(s_1 + \epsilon, t) \in R_B$.

Proof. By Lemma 2.1.6, $I \times \{t\}$ is disjoint from $(R_X \cup R_x) \cup (R_Y \cup R_y)$. By Lemma 2.1.4, a neighborhood of $(0, t)$ (resp. $(1, t)$) in $[0, 1] \times \{t\}$ is contained in $R_A \cup R_a$ (resp. $R_B \cup R_b$). For an $s \in (0, 1)$, if $(s, t)$ is contained in $R_a$ or $R_b$, then $(s, t)$ is contained in $R_x$ or $R_y$, a contradiction. Hence for a small $\epsilon > 0$, $(\epsilon, t)$ (resp. $(1 - \epsilon, t)$) is contained in $R_A$ (resp. $R_B$). Let $s_1 = \sup\{s | [0, s] \times \{t\} \in R_A \cup R_a\}$ and $s_0 = \sup\{s < s_1 | (s, t) \in R_A\}$. Then by Lemma 2.1.3, $s_0 \neq s_1$, and it is clear that the conclusion of Lemma 2.1.7 holds.

Then we apply the argument of the proof of [21, Lemma 2.1] to obtain such $Q'$ from $Q_t$ by an ambient isotopy whose support is contained in $f^{-1}([s_0 + \epsilon, s_1 - \epsilon])$ that satisfies the following conditions.

Conditions 2.1.8. Let $Q^* = f^{-1}([s_0 + \epsilon, s_1 - \epsilon]) \cap Q'$. Then,

1. at each $s \in (s_0, s_1)$, $Q^*$ is transverse to each $P_s$, except for finitely many critical levels $x_1, \ldots, x_n \in (s_0, s_1)$;
2. at each critical level $x_i$, $Q^*$ is transverse to $P_{x_i}$ except for a saddle or circle tangency, as shown in Figure 2.1(a) of [21];
3. at each regular level $s \in (s_0, s_1)$, each component of $P_s \cap Q^*$ is a simple closed curve which is essential in both $P_s$ and $Q^*$.

In the remainder of this paper, we abuse notation by denoting $Q$ for $Q'$.

2.1.3 Sweep-outs induced from $(g, b)$-bridge decompositions

Let $L$ be a link in a closed orientable 3-manifold $M$. Suppose $(A, \tau_A) \cup_P (B, \tau_B)$ is a $(g, n)$-bridge decomposition for $(M, L)$. From the definition of a spine, one
can construct a map \( f : M \rightarrow [-1, 1] \) such that \( f^{-1}(-1) \) is a spine of \((A, \tau_A)\), \( f^{-1}(1) \) is a spine of \((B, \tau_B)\) and \( f^{-1}(s) \) is a surface \( L \)-parallel to the bridge surface \( P \) for each \( s \in (-1, 1) \). This map is called a \textit{sweep-out} induced from \((A, \tau_A) \cup_P (B, \tau_B)\). For each \( s \in (-1, 1) \), we put \( P_s = f^{-1}(s) \), \( A_s = f^{-1}([-1, s]) \) and \( B_s = f^{-1}([s, 1]) \).
2.2 An estimation of Hempel distance by using Reeb Graph

We continue with subsection 2.1.2. Particularly, let \( s_0, s_1 \) be as in Lemma 2.1.7, and \( Q^* \) as in Conditions 2.1.8. Define the equivalence relation \( \sim \) on points on \( Q^* \) by \( x \sim y \) whenever \( x, y \) are in the same component of a level set of \( f|_{Q^*} : Q^* \to [s_0 + \epsilon, s_1 - \epsilon] \). The Reeb graph corresponding to \( f|_{Q^*} \) is the quotient space of \( Q^* \) by the relation \( \sim \). Then \( G \) denotes the Reeb graph corresponding to \( f|_{Q^*} \). Note that \( G \) is a finite 1-complex such that the edges of \( G \) come from annuli in \( Q^* \) fibered by level loops, and that the valency two vertices correspond to circle tangencies, the valency three vertices correspond to saddle tangencies, and the valency one vertices correspond to components of \( \partial Q^* \). In particular, if a valency one vertex corresponds to a component of \( f^{-1}(s_0 + \epsilon) \cap Q \) (resp. \( f^{-1}(s_1 - \epsilon) \cap Q \)), then it is called a \( \partial_-\)vertex (resp. \( \partial_+\)vertex). The union of \( \partial_-\)vertices (resp. \( \partial_+\)vertices) is denoted by \( \partial_- G \) (resp. \( \partial_+ G \)). Let \( f^* : G \to [s_0 + \epsilon, s_1 - \epsilon] \) be the function induced from \( f|_{Q^*} \). Note that for each \( s \in (s_0 + \epsilon, s_1 - \epsilon) \), \( f^{-1}(s)(\subset G) \) consists of a finite number of points corresponding to the components of \( P_s \cap Q^* \).

In [13], a method of estimating Hempel distance by using Reeb graph is given. We note that the Reeb graphs in [13] are slightly different from the above Reeb graphs, since circle tangencies did not come to appear in [13]. However it is easy to see that the arguments in [13] work in the setting of this paper. We introduce it here.

Let \( G, \partial_G, f^* \) be as above. We assign a positive integer to each edge of \( G \) according to the following steps. Let \( w_1, \ldots, w_k \) be the vertices of \( G \) which are not \( \partial \)-vertices. We suppose that \( w_1, \ldots, w_k \) are positioned in this order from the left, i.e., \( f^*(w_1) < f^*(w_2) < \cdots < f^*(w_k) \).

Now we define Steps 0, 1 and 2 inductively for assigning positive integers to the edges of \( G \).

**Step 0.** We assign 1 to every edge adjacent to \( \partial_- G \).

**Step 1.** Suppose that there is a valency two vertex \( w_i \) adjacent to edges \( e_l, e_r \) such that \( e_l \) has already been assigned and \( e_r \) has not been assigned yet. Then we assign the same integer as that of \( e_l \) to \( e_r \). We apply this assignment as
much as possible.

In our assigning process, we will repeat applications of Steps 1 and 2. Before describing Step 2, we will give a general condition that the assignments have in the process. Suppose that we finish Step 1 in repeated applications of Steps 1 and 2. At this stage, either every edge of $G$ is assigned exactly one integer, or there is a unique vertex $w_i$ such that there is an unassigned edge adjacent to $w_i$, and that each edge of $G$ containing a point $x$ with $f^*(x) < f^*(w_i)$ has already been assigned exactly one integer. Then we suppose that the assigned integers satisfy the following condition (*). (Note that the conditions are clearly satisfied after Steps 0 and 1.)

(*) For a small $\epsilon > 0$, let $L_i$ be the set of the edges of $G$ each containing a point $x$ with $f^*(x) = f^*(w_i) - \epsilon$. Then it satisfies one of the following conditions:

1. All of the elements of $L_i$ are assigned with the same integer, say $n$.
2. The set of the integers assigned to the elements of $L_i$ consists of consecutive two integers, say $n - 1$ and $n$.

**Step 2.**

1. Suppose that the vertex $w_i$ satisfies the condition (1). Then we assign $n+1$ to the unassigned edge(s) adjacent to $w_i$.
2. Suppose that the vertex $w_i$ satisfies the condition (2). Then we assign $n$ to the unassigned edge(s) adjacent to $w_i$.

After finishing Step 2, we apply Step 1. Here we note that there are no multiple assignments. If all of the edges are assigned integers, then we are done. Suppose there is an unassigned edge. Then there is a unique vertex $w_j$ such that there is an unassigned edge adjacent to $w_j$, and that each edge of $G$ that contains a point $y$ such that $f^*(y) < f^*(w_j)$ has already been assigned. Then we can show that the assignments at the current stage, also satisfies (*) (for the proof, see Lemma 7.1 of [13]). And by repeating the processes, we finally assign an integer to each edge of $G$.

The next theorem is proved as in the proof of [13].

**Theorem 2.2.1.** Let $P, Q$ and $G$ be as above. Let $n$ be the minimum of the integers assigned to the edges adjacent to $\partial_+ G$. Then the distance $d(P)$ is at most $n + 1$. 

24
Figure 2.6: assigning an integer to each edge of $G$
2.3 Proof of Theorem 1.1.1

Let $P, Q, Q^*, G, \partial \pm G, f, f^*$ be as in Section 2.2. In this section, we suppose that $P$ and $Q$ are genus $g$ Heegaard surfaces and $d(P) = 2g$, and then we consider the Reeb graph $G$ derived from $Q^*$.

Since $Q$ is connected, there is a component of $Q^*$, say $\hat{Q}^*$, whose Reeb graph contains a $\partial_-$-vertex and a $\partial_+$-vertex. Let $\hat{G}^*$ be the Reeb graph corresponding to $\hat{Q}^*$.

**Claim 2.3.1.** The Reeb graph $\hat{G}^*$ contains exactly $2g - 2$ valency three vertices, and each component of $\text{cl}(Q \setminus \hat{Q}^*)$ is an annulus.

**Proof.** By 3 of Condition 2.1.8, each component of $\partial \hat{Q}^*$ is essential on $Q$. Hence by Euler characteristic argument, we see that $\hat{Q}^*$ contains at most $2g - 2$ saddles. By the rule in the assigning process, for each valency three vertex $v$, the difference in the integers assigned to the edges adjacent to $v$ is at most one. Hence the integer assigned to the edges of $\hat{G}^*$ containing $\partial_+$-vertex is at most $2g - 1$. This fact together with Theorem 2.2.1 and the assumption $d(P) = 2g$ show that each edge containing $\partial_+$-vertex is assigned $2g - 1$. Moreover by the rule in the assigning process, we see that $\hat{Q}^*$ contains exactly $2g - 2$ saddles. Hence $\chi(Q \setminus \hat{Q}^*) = \chi(Q) - \chi(\hat{Q}^*) = 0$, and this implies that each component of $\text{cl}(Q \setminus \hat{Q}^*)$ is an annulus. $\square$

Let $G$ be any path in the Reeb graph $\hat{G}^*$ joining a $\partial_-$-vertex and a $\partial_+$-vertex.

**Claim 2.3.2.** The path $G$ contains all of the valency three vertices in $\hat{G}^*$.

**Proof.** Assume that $G$ does not contain all of the valency three vertices. Then by Claim 2.3.1, $G$ contains at most $2g - 3$ valency three vertices. This fact together with the rule in the assigning process show that the edge of $G$ adjacent to an $\partial_+$-vertex is assigned an integer less than $2g - 1$. Then Theorem 2.2.1 show that $d(P) < 2g$, a contradiction. Hence this claim holds. $\square$

Let $v_1, \ldots, v_{2g-2}$ be the valency three vertices contained in $G$. By the proof of Claim 2.3.2, we see that by changing subscripts if necessary, we may suppose for each $i$, the set of the integers assigned to the edges adjacent to $v_i$ consists of consecutive two integers $\{i, i + 1\}$. 

26
Claim 2.3.3. \( f^*(v_1) < f^*(v_2) < \cdots < f^*(v_{2g-2}) \).

Proof. Suppose that there exists \( i \) such that \( f^*(v_i) > f^*(v_{i+1}) \), then there is a point \( x \) in the path \( \mathcal{G} \), which joins \( v_{i+1} \) and a \( \partial_- \)-vertex such that \( f^*(v_{i+1}) < f^*(x) < f^*(v_i) \). Note that by Claim 2.3.2, the edge containing \( x \) is assigned at least \( i + 2 \). Then there is a point \( x' \) in the path \( \mathcal{G} \), which joins a \( \partial_- \)-vertex and \( v_i \) such that \( f^*(x) = f^*(x') \). By Claim 2.3.2, the integer assigned to the edge containing \( x' \) is at most \( i \), contradicting condition (*). \( \square \)

By Claim 2.3.1 and the assumption \( d(P) = 2g \), we see that \( \hat{\mathcal{Q}}^* \) is the unique component whose Reeb graph contains both a \( \partial_- \)-vertex and a \( \partial_+ \)-vertex.

![Figure 2.7: The Reeb graph \( \hat{\mathcal{G}}^* \)](image)

Claim 2.3.4. For \( v_i \) (\( i = 1, \ldots, 2g - 2 \)), if \( i \) is odd, then the number of edges assigned \( i \) adjacent to \( v_i \) is exactly two. If \( i \) is even, then the number of edges assigned \( i \) adjacent to \( v_i \) is exactly one.

Proof. By the fact that each component of \( cl(Q \setminus \hat{\mathcal{Q}}^*) \) is an annulus (Claim 2.3.1) and 3 of Conditions 2.1.8, we see that each component of \( Q^* \) other than \( \hat{Q}^* \) is an annulus. This fact together with Theorem 2.2.1 and the assumption \( d(P) = 2g \) show that both of the boundary components of each annulus are contained in \( f^{-1}(s_0 + \epsilon) \) or \( f^{-1}(s_1 - \epsilon) \). This shows that the number of edges containing \( \partial_- G \) is even, and the number of \( \partial_- \)-vertices contained in \( \hat{G}^* \) is even. This together with the fact that \( v_1 \) is the only vertex which is adjacent to edges assigned 1, show that the number of edges assigned 1 in \( \hat{G}^* \) adjacent to \( \partial_- G \) is exactly two. This implies that the number of edges assigned 1 in \( \hat{G}^* \) adjacent to \( v_1 \) is two and the number of edges assigned 2 adjacent to \( v_1 \) is one. This
implies that the number of edges assigned 2 adjacent to \( v_2 \) is exactly one. By this fact and the fact that for each \( i \), the set of the integers assigned to the edges adjacent to \( v_i \) consists of consecutive two integers \( \{i, i+1\} \), we see that the remaining two edges are assigned 3. By repeating the arguments, we have this claim.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.8.png}
\caption{\( i \) is odd}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.9.png}
\caption{\( i \) is even}
\end{figure}

We say that \( v_i \) is in a normal position, if \( v_i \) satisfies the following,

For each edge \( e \) adjacent to \( v_i \), if \( e \) is assigned \( i \) (resp. \( i + 1 \)), then for each \( x \in e \), \( f^*(x) < f^*(v_i) \) (resp. \( f^*(x) > f^*(v_i) \)).

**Claim 2.3.5.** For each \( v_i \), there is an ambient isotopy \( \varphi_1^{(i)} \) of \( M \) whose support is contained in \( f^{-1}(P \times [f^*(v_i) - \epsilon, f^*(v_i) + \epsilon]) \) such that \( \varphi_1^{(i)}(Q^*) \) satisfies Conditions 2.1.8, hence, the Reeb graph of \( \varphi_1^{(i)}(Q^*) \) is defined, where the Reeb graph contains exactly one valency three vertex between the levels \( f^*(v_i) - \epsilon \) and \( f^*(v_i) + \epsilon \), which is in a normal position.

**Proof.** Suppose that \( v_i \) is not in a normal position. Then the possible patterns of assigned edges adjacent to \( v_i \) are shown in Figure 2.10 (1)--(6).

28
We consider the pattern (1). The component of \( f^{-1}(P \times [f^*(v_i) - \epsilon, f^*(v_i) + \epsilon]) \cap Q^* \) containing the critical point of \( f|_{Q^*} \) is a wedge of two circles \( c_1, c_2 \) on \( Q^* \). Here, \( c_1 \) corresponds to the edge assigned \( i + 1 \). Now we take a narrow annulus \( A \) such that \( A \cap Q^* = \partial A \cap Q^* = c_1 \), \( f|_{A} \) has no critical point, and \( f(\partial A \setminus c_1) = f^*(v_i) + \epsilon/2 \). Then we push \( c_1 \) along \( A \) to deform \( Q^* \) as in Figure 2.11. By this figure, we see that this ambient isotopy gives \( \varphi^{(i)}_i \).

The pattern (4) can be treated in a similar argument as in the pattern (1).

The patterns (2), (5) can be treated by two successive applications of the above arguments as described in Figure 2.12. Details the left to the reader.

The patterns (3), (6) can be treated by three successive applications of the above arguments as described in Figure 2.13. Details the left to the reader. \( \square \)
Figure 2.12: The patterns (2) and (5)

Figure 2.13: The patterns (3) and (6)

Here, for simplicity, we use $Q^*$ for $\varphi_1^4(\varphi_2^2(\cdots(\varphi_1^{2g-2}(Q^*))\cdots))$.

Claim 2.3.6. There is an ambient isotopy $\psi_1^{(1)}$ of $M$ whose support is contained in $f^{-1}(P \times [s_0 + \epsilon, f^*(v_2)])$ such that $\psi_1^{(1)}(Q^*)$ satisfies Conditions 2.1.8, hence, the Reeb graph of $\psi_1^{(1)}(Q^*)$ is defined, and it satisfies the following. There is exactly one valency three vertex in the levels $(s_0 + \epsilon, f^*(v_2))$, which has the level $f^*(v_1)$ and is in a normal position, and every edge which is assigned 1 is contained in between the levels $s_0 + \epsilon$ and $f^*(v_1)$.

Proof. Let $\mathcal{P}_1, \mathcal{P}_2$, be the path in $\mathcal{C}^*$ each of which joins a $\partial_-$-vertex and $v_1$. We consider the subset $\mathcal{Q}_i$ of $\mathcal{P}_i$ consisting of the point $x$ with $f^*(x) \geq f^*(v_1)$. Since $v_1$ is in a normal position (Claim 2.3.5), $\mathcal{Q}_i$ is a union of subarcs contained in $\text{Int}(\mathcal{P}_i)$. By Condition (*) in Section 2.2, we see that $\mathcal{Q}_i$ is contained between the levels $f^*(v_1)$ and $f^*(v_2)$. Note that the preimage of each arc is an annulus properly embedded in the product region $f^{-1}([f^*(v_1), f^*(v_2)])$, whose boundary is contained in the boundary component of $f^{-1}(f^*(v_1))$. This shows that there is an ambient isotopy of $M$ whose support is contained in a small
neighborhood of $f^{-1}([f^*(v_1), f^*(v_2)])$, which pushes the annuli corresponding to $Q_1 \cup Q_2$ out of $f^{-1}([f^*(v_1), f^*(v_2)])$. Note that this ambient isotopy does not affect the vertex $v_1$. Hence we obtained a desired ambient isotopy. □

![Figure 2.14: ambient isotopy $\psi^{(1)}_k$](image)

Here, for simplicity, we use $Q^*$ for $\psi^{(1)}_1(Q^*)$. By using similar arguments as in the proof of Claim 5.6, we have the following.

**Claim 2.3.7.** There is an ambient isotopy $\psi^{(2)}_l$ of $M$ whose support is contained in $f^{-1}(P \times [f^*(v_1), f^*(v_3)])$ which satisfies Conditions 2.1.8, hence, the Reeb graph of $\psi^{(2)}_l(Q^*)$ is defined, and it satisfies the following. There is exactly one valency three vertex in the levels $(f^*(v_1), f^*(v_2))$, which has the level $f^*(v_2)$ and is in a normal position, and every edge which is assigned 2 is contained in $[f^*(v_1), f^*(v_2)]$.

![Figure 2.15: ambient isotopy $\psi^{(2)}_k$](image)

Then we successively apply similar argument to $v_3, v_4, \ldots, v_{2g-2}$, and we finally obtain the next claim.
Claim 2.3.8. We may suppose that $Q^*$ satisfies the following:

1. each $v_i$ in $G^*$ is in a normal position.
2. for each $i$ ($1 \leq i \leq 2g - 3$), each edge assigned $i + 1$ is contained in between the levels $f^*(v_i)$ and $f^*(v_{i+1})$. The edge assigned 1 is contained in the levels $s_0 + \epsilon$ and $f^*(v_1)$, and the edge assigned $2g - 1$ is contained in the levels $f^*(v_{2g-2})$ and $s_1 - \epsilon$.

Claim 2.3.9. There is an ambient isotopy $\Phi_t$ of $M$ such that $\Phi_1(Q^*)$ satisfies Condition 2.1.8, hence, the Reeb graph of $\Phi_1(Q^*)$ is defined, and it satisfies the following. For each $i$ ($1 \leq i \leq 2g - 3$), each edge assigned $i + 1$ joins $v_i$ and $v_{i+1}$, each edge assigned 1 joins a $\partial_-$-vertex and $v_1$, and each edge assigned $2g - 1$ joins $v_{2g-2}$ and a $\partial_+$-vertex.

Proof. By Claim 2.3.8, we see that for each $i$ ($1 \leq i \leq 2g - 3$), each component of the subset of $Q^*$ between the levels $f^*(v_i)$ and $f^*(v_{i+1})$, or between the levels $s_0 + \epsilon$ and $f^*(v_1)$, or between the levels $f^*(v_1)$ and $s_1 + \epsilon$, is an annulus (with pinches introduces by saddles) properly embedded in the corresponding product regions $f^{-1}([f^*(v_i), f^*(v_{i+1})])$, or $f^{-1}([s_0 + \epsilon, f^*(v_1)])$, or $f^{-1}([f^*(v_{2g-2}), s_1 - \epsilon])$. Hence by using ambient isotopy, we can straighten each annulus with the boundary components fixed so that no critical point exists in the interior of the annulus. The composition of the above ambient isotopes gives $\Phi_t$.

Here, for simplicity, we use $Q^*$ for $\Phi_1(Q^*)$.

Figure 2.16: The Reeb graph of $\hat{Q}^*$

Completion of the proof

We have obtained $Q$ with $Q^*$ satisfies the conditions in Claim 2.3.9. Recall that $\hat{Q}^*$ is the component of $Q^*$ joining a $\partial_-$-vertex and a $\partial_+$-vertex.
By Claim 2.3.1 and 3 of Conditions 2.1.8, we see that each component of $Q^*$ other than $\hat{Q}^*$ is an annulus properly embedded in the product region $f^{-1}([s_0 + \epsilon, s_1 - \epsilon])$. Then by the proof of Claim 2.3.6, we can push such annulus out of the product region. Hence we may suppose that $\hat{Q}^* = Q^*$, and this implies that $\text{cl}(Q \setminus \hat{Q}^*)$ consists of two annuli, say $A_-, A_+$ such that $A_-$ is properly embedded in the handlebody $f^{-1}([0, s_0 + \epsilon])$, and $A_+$ is properly embedded in the handlebody $f^{-1}([s_1 - \epsilon, 1])$. Assume that $A_-$ is compressible in the handlebody $f^{-1}([0, s_0 + \epsilon])$. Then by compressing $A_-$, we see that each component of $\partial A_-$ bounds a disk in the handlebody, and hence has distance 0 in the handlebody. This together with the saddles in $Q^*$ shows that $d(P)$ is at most $2g - 1$, a contradiction. Hence $A_-$ is incompressible in the handlebody.

Then we have the following two cases.

Case 1. $A_-$ is not essential in the handlebody (i.e., that is, $A_-$ is $\partial$-parallel in the handlebody).

In this case, we can isotope $A_-$ with fixing $\partial A_-$ so that $A_-$ is contained in a small collar neighborhood of the boundary of the handlebody, which is the union of level surfaces. Then it is easy to see that we can further isotope $A_-$ in the collar neighborhood so that the critical points of $f|_{A_-}$ consists of one minimal point, and one saddle point.

Case 2. $A_-$ is essential in the handlebody.

In this case, we can $\partial$-compress $A_-$ to obtain a meridian disk, say $D_-$, in the handlebody. Then we can take a spine $\Sigma_-$ of the handlebody such that $\Sigma_-$ intersects $D_-$ transversely in a single point. Then we retake $f$ with respect to $\Sigma_-$ so that $\Sigma_- \cap D_-$ is the unique critical point (which is obviously minimal) of $f|_{D_-}$. Note that $A_-$ can be recovered from $D_-$ by adding a band. We may suppose that the band is contained in a collar neighborhood as in Case 1. Hence we may suppose that this band contains exactly one critical point, which is a saddle. Hence the critical points of $f|_{A_-}$ consists of one minimal point and one saddle point.

In either case we have shown that the critical points of $f|_{A_-}$ consists of one minimal point and one saddle point. The annulus $A_+$ properly embedded in the handlebody $f^{-1}([s_1 - \epsilon, 1])$ is treated in same manner. These fact together with Claim 2.3.9 completes the proof of Theorem 1.1.1.
2.4 Proofs of Theorem 1.1.2 and Corollary 1.1.3

Let $B^3$ be a 3-ball, and $\tau$ a set of trivial $n$-arcs in $B^3$ with $n \geq 3$.

**Lemma 2.4.1.** Let $D$ be a $\gamma$-compressing disk in $B^3$. Then for a $\tau$-essential simple closed curve $\ell \subset \partial(B^3 \setminus \partial\tau)$ which is disjoint from $\partial D$, $d((D(B^3 \setminus \tau), \ell)) \leq 1$.

*Proof.* If $D \cap \tau = \emptyset$, then clearly Lemma 2.4.1 holds. Suppose $D \cap \tau$ consists of one point. Then $D$ separates $(B^3, \tau)$ into two components $(B^3_1, \tau_1)$ and $(B^3_2, \tau_2)$ where $\ell \subset \partial B^3_1 \setminus \partial\tau_1$. Since $D$ is a $\tau$-compressing disk, $\tau_2$ consists of at least two components. Hence there is a compressing disk $D' \subset B^3_2$ for $\partial B^3_2 \setminus \partial\tau_2$. Further, by $\tau_2$-isotopy of $B^3_2$, we may suppose that $D'$ is disjoint from the image of $D$. Hence we may regard $D'$ a compressing disk in $B^3 \setminus \tau$. Since $\ell$ is disjoint from $D'$, we have $d(\ell, \partial D') \leq 1$. \qed

![compression disk](image)

Figure 2.17:

Let $L$ be a link in $S^3$, $(A, \tau_A) \cup_P (B, \tau_B)$ a $(0, n_1)$-bridge decomposition ($n_1 \geq 3$) for $(S^3, L)$, and $f$ a sweep-out induced from $(A, \tau_A) \cup_P (B, \tau_B)$. Let $\pi_P$ be the projection map from $P \times (-1, 1)$ to $P$.

Suppose that $Q$ is a $(0, n_2)$-bridge surface ($n_2 \geq 3$) for $(S^3, L)$ which is not equivalent to $P$. Then, by [17, Theorem 3.1 and Lemma 4.3], $Q$ can be $L$-isotoped so that $f|_Q$ is Morse, and every $s \in (-1, 1)$, $P_s \cap Q$ contains a curve that is $L$-essential in $P_s$. Moreover, as in the proof of [17, Theorem 4.2], either $d(P, L) \leq 1$ or there exists an interval $[s_-, s_+]$, where $s_- < s_+$ are critical values for $f|_Q$ such that

1. for every $s \in (s_-, s_+)$, each component of $P_s \cap Q$ which is $L$-essential in $P_s$ does not bound a disk in $Q$, and

34
2. for a small $\epsilon$, $P_{s_- - \epsilon} \cap Q$ contains a curve that is $L$-essential in $P_{s_- - \epsilon}$ and bounds a disk in $A_{s_- - \epsilon}$, and $P_{s_+ + \epsilon} \cap Q$ contains a curve that is $L$-essential in $P_{s_+ + \epsilon}$ and bounds a disk in $B_{s_+ + \epsilon}$.

Hence we have: (1) if $d(P, L) \leq 2$, there exists an interval $[s_-, s_+]$ satisfying the condition (1) and (2). We show that in (1), the conclusion can be improved if $d(P, L) > 2$. Namely:

Lemma 2.4.2. Let $Q$ be as above. If $d(P, L) > 2$, there exists a subinterval $[s', s'_+] \subset [s_-, s_+]$, where $s' < s'_+$ are critical values for $f|_Q$ such that

(i) for every $s' \in (s', s'_+]$, each component of $P_{s'} \cap Q$ which is $L$-essential in $P_{s'}$ does not bound an $L$-disk in $Q$, and

(ii) for a small $\epsilon$, $P_{s'_- - \epsilon} \cap Q$ contains a curve that is $L$-essential in $P_{s'_- - \epsilon}$ and bounds an $L$-disk in $A_{s'_- - \epsilon}$, and $P_{s'_+ + \epsilon} \cap Q$ contains a curve that is $L$-essential in $P_{s'_+ + \epsilon}$ and bounds an $L$-disk in $B_{s'_+ + \epsilon}$.

Proof. Suppose that for every $s \in (s_-, s_+)$, there exists a component of $P_s \cap Q$ which is $L$-essential in $P_s$ bounds a $L$-disk in $Q$. Then there exists a critical value $s^*$ (possibly $s^* = s_-$ or $s_+$) such that for a small $\epsilon$, $P_{s^* - \epsilon} \cap Q$ contains a curve bounding a $L$-disk $D_A$ in $A_{s^* - \epsilon}$, and $P_{s^* + \epsilon} \cap Q$ contains a curve bounding a $L$-disk $D_B$ in $B_{s^* + \epsilon}$. Note that $d(\pi_P(\partial D_A), \pi_P(\partial D_B)) \leq 1$, where $\pi_P$ be the projection map from $P \times (-1, 1)$ to $P$. Hence by regarding $D_A$ as $D$ and $\pi_P(\partial D_B)$ as in $\ell$ in Lemma 2.4.1, we have $d(\partial D_A \setminus \tau_A, \pi_D(\partial D_B)) \leq 1$. Analogously we have $d(\partial D_B \setminus \tau_B, \pi_D(\partial D_B)) \leq 1$. These together with a triangle inequality $d_P(\partial D_A \setminus \tau_A, \partial D_B \setminus \tau_B)) \leq d(\partial D_A \setminus \tau_A, \pi_P(\partial D_B)) + d(\pi_P(\partial D_B), \partial D_B) \setminus \tau_B) \leq 1$. Hence, Lemma 2.4.2 holds. □

Proof of Theorem 1.1.2. Let $P, Q, f$ be as above. By Lemma 2.4.2, either (I)$d(P, L) \leq 2$ or (II)there exists an interval $[s'_-, s'_+]$, where $s'_- < s'_+$ are critical values for $f|_Q$ satisfying (i) and (ii) of Lemma 2.4.2. If $d(P, L) \leq 2$, then since $n_2 \leq 3$, the conclusion of Theorem 1.1.2 holds. Hence we consider the case (II). Let $C$ be the union of the components of $P_{s'_- + \epsilon} \cap Q$ and $P_{s'_+ - \epsilon} \cap Q$ which are $L$-essential on $Q$. Since $Q$ is connected, there is a component, say $Q'$, of $Q \setminus C$ such that $\partial(Q') \cap P_{s'_+ - \epsilon} \neq \emptyset$ and $\partial(Q') \cap P_{s'_- - \epsilon} \neq \emptyset$. Note that $\chi(Q' \setminus L) \geq \chi(Q \setminus L) + 2$. Let $c_-$ (resp. $c_+$) be a component of $cl(Q') \cap P_{s'_- + \epsilon}$
(resp. $cl(Q') \cap P_{x^+}$). Hence by using arguments as in the proof of [17, Theorem 4.2], $d(\pi_P(c_-), \pi_P(c_+)) \leq -\chi(Q' \setminus L)$, where $\pi_P$ be the projection map from $P \times (-1, 1)$ to $P$. By (ii) of Lemma 2.4.2, $\pi_P(c_-)$ (resp. $\pi_P(c_+)$) is disjoint from $L$-compressing disk in $A$ (resp. $B$). By Lemma 2.4.1, $d(D(A \setminus \tau_A), \pi(c_-)) \leq 1$ and $d(\pi(c_+), D(B \setminus \tau_B)) \leq 1$. Hence, we have

$$d(P, L) \leq d(D(A \setminus \tau_A), \pi_P(c_-)) + d(\pi_P(c_-), \pi_P(c_+)) + d(\pi(c_+), D(B \setminus \tau_B)) \leq 1 - \chi(Q' \setminus L) + 1 = -\chi(Q \setminus L).$$

This completes the proof of Theorem 1.1.2.

Proof of Corollary 1.1.3. Let $Q$ be a minimal bridge sphere for a link $L$. Suppose that $Q$ not equivalent to $P$. Then, by Theorem 1.1.2, $d(P, L) \leq |Q \cap L| - 2 = |P \cap K| - 2$, a contradiction.
Chapter 3
Heegaard splitting with distance exactly \( n \)

3.1 A pair of curves with distance exactly \( n \)

In this section, we construct pairs of curves with distance exactly \( n \).

3.1.1 Subsurface projection maps

Let \( \mathcal{P}(V) \) denote the power set of a set \( V \). Suppose that \( X \) is an essential non-simple subsurface of \( S \). We call the composition \( \pi_0 \circ \pi_A \) of maps \( \pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X)) \) and \( \pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X)) \) a subsurface projection if they satisfy the following: for a vertex \( \alpha \), take a representative \( \alpha \) so that \( |\alpha \cap X| \) is minimal, where \( |\cdot| \) is the number of connected components. Then

- \( \pi_A(\alpha) \) is the set of all isotopy classes of the components of \( \alpha \cap X \),

- \( \pi_0(\{\alpha_1, \ldots, \alpha_n\}) \) is the union for all \( i = 1, \ldots, n \) of the set of all isotopy classes of the components of \( \partial N(\alpha_i \cup \partial X) \) which are essential in \( X \), where \( N(\alpha_i \cup \partial X) \) is a regular neighborhood of \( \alpha_i \cup \partial X \) in \( X \).

We say that \( \alpha \) misses \( X \) (resp. \( \alpha \) cuts \( X \)) if \( \alpha \cap X = \emptyset \) (resp. \( \alpha \cap X \neq \emptyset \)).

Lemma 3.1.1. Let \( X \) be as above. Let \( [\alpha_0, \alpha_1, \ldots, \alpha_n] \) be a path in \( \mathcal{C}(S) \) such that every \( \alpha_i \) cuts \( X \). Then \( \text{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n \).

Proof. Since \( d_S(\alpha_i, \alpha_{i+1}) = 1 \) and every \( \alpha_i \) cuts \( X \), we have
diam_{AC(X)}(\pi_A(\alpha_i), \pi_A(\alpha_{i+1})) \leq 1
for every \(i = 1, 2, \ldots, n-1\). By [24, Lemma 2.2], we have
\[
diam_X(\pi_0(\pi_A(\alpha_i)), \pi_0(\pi_A(\alpha_{i+1}))) \leq 2.
\]
Hence, since \(diam_{AC(X)}(\pi_A(\alpha_0), \pi_A(\alpha_n)) \leq n\), we have
\[
diam_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) = diam_X(\pi_0(\pi_A(\alpha_0)), \pi_0(\pi_A(\alpha_n)))
\leq 2n.
\]

Remark 3.1.2. If \(X\) is an essential subsurface of \(S\) with at least two components, then for any pair of curves \(\alpha, \alpha'\) on \(S\) we have \(diam_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2\).
To be precise, let \(X_1\) be one of the components of \(X\), and \(X_2\) the union of the others. Let \(a\) and \(a'\) be elements of \(\pi_X(\alpha)\) and \(\pi_X(\alpha')\), respectively. If both \(a\) and \(a'\) are contained in \(X_i\) for some \(i = 1, 2\), say \(X_1\), then we can find a curve on \(X_2\) that is disjoint from \(a \cup a'\), which implies \(d_X(a, a') \leq 2\). If \(a \subset X_1\) and \(a' \subset X_2\) (or \(a \subset X_2\) and \(a' \subset X_1\)), we have \(d_X(a, a') \leq 1\). Thus \(d_X(a, a') \leq 2\) for any pair of elements \(a \in \pi_X(\alpha)\) and \(a' \in \pi_X(\alpha')\), and hence we have \(diam_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2\).

3.1.2 A pair of curves with distance exactly \(n\)

Let \(S\) be a closed orientable surface with genus greater than 1. We first prove the following two propositions.

Proposition 3.1.3. For an even positive integer \(n(\geq 4)\), let \([\alpha_0, \alpha_1, \ldots, \alpha_n]\) be a path in \(C(S)\) satisfying the following.

\((H0)\) \(\alpha_{n-2}\) is non-separating,
\((H1)\) \([\alpha_0, \ldots, \alpha_{n-2}]\) and \([\alpha_{n-2}, \alpha_{n-1}, \alpha_n]\) are geodesics in \(C(S)\),
\((H2)\) \(diam_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_{n})) \geq 4n\), where \(X_{n-2} = Cl(S \setminus N(\alpha_{n-2}))\).

Then \([\alpha_0, \alpha_1, \ldots, \alpha_n]\) is a geodesic in \(C(S)\).

Proof. Let \([\beta_0, \beta_1, \ldots, \beta_m]\) be a geodesic in \(C(S)\) such that \(\beta_0 = \alpha_0\), \(\beta_m = \alpha_n\). Then \(m \leq n\).
Claim 3.1.4. $\beta_j = \alpha_{n-2}$ for some $j \in \{0, 1, \ldots, m\}$.

Proof. Assume on the contrary that $\beta_j \neq \alpha_{n-2}$ for any $j$. By Remark 3.1.2, every $\beta_j$ cuts $X_{n-2}$ (because $\alpha_{n-2}$ is non-separating). By Lemma 3.1.1, we have $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m)) \leq 2m$. Since $[\alpha_0, \alpha_1, \ldots, \alpha_{n-2}]$ is a geodesic, no $\alpha_i$ ($0 \leq i \leq n-3$) is isotopic to $\alpha_{n-2}$. Hence each $\alpha_i$ ($0 \leq i \leq n-3$) cuts $X_{n-2}$ for any $i \in \{0, \ldots, n-4\}$. By Lemma 3.1.1, $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_{n-4})) \leq 2(n-4) < 2n$. Hence,

$$\begin{align*}
\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) & \leq \text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_0)) \\
& \quad + \text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0), \pi_{X_{n-2}}(\alpha_n)) \\
& < 2n + \text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0), \pi_{X_{n-2}}(\beta_m)) \\
& \leq 2n + 2m
\end{align*}$$

Meanwhile, by the hypothesis (H2), we have $\text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) \geq 4n$. Hence $4n \leq \text{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}), \pi_{X_{n-2}}(\alpha_n)) < 2m + 2n$, and we have $m > n$, a contradiction. \qed

By Claim 3.1.4 and the hypothesis (H1), we have the equalities

$$\begin{align*}
j &= d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n - 2, \\
m - j &= d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.
\end{align*}$$

By combining the above inequalities, we have $m = n$. Hence $d_S(\alpha_0, \alpha_n) = n$. \qed

Proposition 3.1.5. For a positive integer $n$, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following.

(H0) $\alpha_{n-1}$ and $\alpha_{n-2}$ are non-separating,

(H1') $[\alpha_0, \ldots, \alpha_{n-1}]$ and $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ are geodesics in $\mathcal{C}(S)$,

(H2') $\text{diam}_S'(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) > 2n$, where $S' = \text{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$.

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Proof. Let $[\beta_0, \beta_1, \ldots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then $m \leq n$. 

39
3.1.6. There exists \( j \in \{0, 1, \ldots, m\} \) such that \( \beta_j = \alpha_{n-2} \) or \( \beta_j = \alpha_{n-1} \).

**Proof.** Suppose that \( \beta_j \neq \alpha_{n-2} \) and \( \beta_j \neq \alpha_{n-1} \) for any \( j \). Then each \( \beta_j \) cuts \( S' \).

Hence, by Lemma 3.1.1, we have \( \text{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) = \text{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) \leq 2m \). On the other hand, by (H2'), \( \text{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) > 2n \), a contradiction. \( \square \)

Suppose \( \beta_j = \alpha_{n-2} \). Then we have the equalities

\[
    j = d_S(\beta_0, \beta_j) = d_S(\alpha_0, \alpha_{n-2}) = n - 2, \\
    m - j = d_S(\beta_j, \beta_m) = d_S(\alpha_{n-2}, \alpha_n) = 2.
\]

By combining the above equalities, we have \( n = m \). Hence \( d_S(\alpha_0, \alpha_n) = n \).

We can use a similar argument for the case when \( \beta_j = \alpha_{n-1} \). This completes the proof of Proposition 3.1.5. \( \square \)

For a given integer \( n \), we construct a geodesic \([\alpha_0, \alpha_1, \ldots, \alpha_n]\) in \( C(S) \), i.e., \( d_S(\alpha_0, \alpha_n) = n \), by using Propositions 3.1.3 and 3.1.5.

**A construction of a concrete example: the case when \( n \) is even**

We first assume that \( n \) is even. Let \( \alpha_0, \alpha_2 \) be essential non-separating simple closed curves on \( S \) which intersect transversely in one point, and let \( \alpha_1 \) be an essential simple closed curve on \( S \) which is disjoint from \( \alpha_0 \cup \alpha_2 \). Let \( X_2 = \text{Cl}(S \setminus N(\alpha_2)) \). Note that \([\alpha_0, \alpha_1, \alpha_2]\) is a geodesic of length two in \( C(S) \).

Choose a homeomorphism \( f_2 : S \to S \) such that \( f_2(N(\alpha_2)) = N(\alpha_2) \) and that \( \text{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n \). This is possible by [23, Proposition 4.6]. Let \( \alpha_3 = f_2(\alpha_1) \) and \( \alpha_4 = f_2(\alpha_0) \). Note that \([\alpha_2, \alpha_3, \alpha_4]\) is a geodesic of length two in \( C(S) \).

We repeat this process to construct a path \([\alpha_0, \alpha_1 \ldots, \alpha_n]\). Namely, for each even \( i \in \{2, 4, \ldots, n-2\} \),

(i-1) \( X_i = \text{Cl}(S \setminus N(\alpha_i)) \),

(i-2) \( f_i : S \to S \) is a homeomorphism such that \( f_i(N(\alpha_i)) = N(\alpha_i) \) and that \( \text{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n \),

(i-3) \( \alpha_{i+1} = f_i(\alpha_{i-1}) \) and \( \alpha_{i+2} = f_i(\alpha_{i-2}) \).
Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $C(S)$.

**Proposition 3.1.7.** For each $k \in \{2, 4, \ldots, n\}$, a path $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ in $C(S)$ is a geodesic.

**Proof.** We prove the proposition by mathematical induction on $k$. It is clear that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $C(S)$. Hence, Proposition 3.1.7 holds for $k=2$.

Assume that $[\alpha_0, \alpha_1, \ldots, \alpha_k]$ is a geodesic in $C(S)$, where $k \in \{2, 4, \ldots, n-2\}$. We note that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]$ is a geodesic in $C(S)$. Furthermore, by the condition (i-2), we have $\text{diam}_{X_k}(\pi_{X_k}(\alpha_{k-2}), \pi_{X_k}(\alpha_{k+2})) \geq 4n > 4k$. Hence, by Proposition 3.1.3, the path $[\alpha_0, \alpha_1, \ldots, \alpha_{k+2}]$ is a geodesic in $C(S)$.

**A construction of a concrete example: the case when $n$ is odd**

Suppose that $n$ is odd. Let $[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$ be a path in $C(S)$ as in the previous subsection, where each $\alpha_i$ is a non-separating curve. Note that $\alpha_{n-3}$ intersects $\alpha_{n-1}$ transversely in one point and is disjoint from $\alpha_{n-2}$. Note also that $\alpha_{n-2}$ is non-separating. It is easy to see that these imply that $\alpha_{n-1} \cup \alpha_{n-2}$ is non-separating. Choose a non-separating essential simple closed curve $\gamma$ on $S$ such that $\gamma \cap \alpha_{n-1} = \emptyset$ and $\gamma$ intersects $\alpha_{n-2}$ transversely in one point. Let $S' = \text{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$. By [23, Proposition 4.6], there exists a homeomorphism $f : S \to S$ such that $f(S') = S'$ and $\text{diam}_{S'}(\pi_{S'}(f(\gamma)), \pi_{S'}(\alpha_0)) > 2n$. Let $\alpha_n = f(\gamma)$. Then, since $\alpha_n \cap \alpha_{n-1} = \emptyset$ and $\alpha_n$ intersects $\alpha_{n-2}$ in one point, we see that $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ is a geodesic in $C(S)$. On the other hand, $[\alpha_0, \ldots, \alpha_{n-1}]$ is also a geodesic in $C(S)$. Hence, by Proposition 3.1.5, the path $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $C(S)$.

We remark that the construction of geodesics introduced in this section works for majority of compact surfaces.
3.2 Heegaard splitting with distance exactly $n$ for each integer $n > 0$

3.2.1 Disk complex for a special compression body

Let $\mathcal{D}(V) (\subset \mathcal{C}(\partial_+ V))$ be the disk complex of a compression body $V$. We have a decomposition $\mathcal{D}(V) = \mathcal{D}_{\text{nonsep}}(V) \sqcup \mathcal{D}_{\text{sep}}(V)$, where $\mathcal{D}_{\text{nonsep}}(V)$ (resp. $\mathcal{D}_{\text{sep}}(V)$) denotes the subset of $\mathcal{D}(V)$ consisting of vertices with representatives bounding non-separating (resp. separating) disks of $V$. In this section, we prove the following proposition.

**Proposition 3.2.1.** Let $V$ be a compression body obtained by adding a 1-handle to $F \times [0, 1]$, where $F$ is a genus-$(g-1)$ closed orientable surface ($g > 1$). Then we have the following.

1. $\mathcal{D}_{\text{nonsep}}(V)$ consists of a point, say $c_0$.

2. For each element $c_\alpha$ of $\mathcal{D}_{\text{sep}}(V)$, there is a 1-simplex in $\mathcal{C}(\partial_+ V)$ joining $c_0$ and $c_\alpha$.  

![Figure 3.3: a genus-3 compression body](image-url)
Remark 3.2.2. In fact, we can see that $A$ is a countable, infinite set and that there is no 1-simplex between $c_\alpha$ and $c_{\alpha'}$, for each pair $\alpha, \alpha' \in A$.

In the remaining of this section, $V$ denotes a compression body obtained by adding a 1-handle to $F \times [0, 1]$, where $F$ is a genus-$(g - 1)$ closed orientable surface $(g > 1)$. Then Proposition 3.2.1 follows from Lemmas 3.2.3 and 3.2.4.

Lemma 3.2.3. Any two non-separating disks are ambient isotopic.

Proof. Let $D$ be the co-core of the the 1-handle attached to $F \times [0, 1]$ and let $D'$ be another non-separating disk in $V$. Assume that $D$ and $D'$ intersect transversely, and $|D \cap D'|$ is minimized up to ambient isotopy class of $D'$.

Suppose $|D \cap D'| = 0$, i.e., $D \cap D' = \emptyset$. Since any disk properly embedded in $F \times [0, 1]$ is boundary parallel, we see that $D \cup D'$ bounds a product region, and hence $D'$ is ambient isotopic to $D$.

Suppose $|D \cap D'| > 0$. By standard innermost disk arguments, we can see that $D \cap D'$ has no loop components. Note that there are at least two components of $D' \setminus (D \cap D')$ which are outermost in $D'$. Take a pair of such outermost components, say $\Delta_1$ and $\Delta_2$, which are the next to each other, i.e., there is a subarc $\beta \subset \partial D'$ such that $\beta \cap \Delta_1$ is an endpoint of $\beta$ and $\beta \cap \Delta_2$ is the other endpoint of $\beta$, and $\beta$ does not intersect any other outermost disk of $D' \setminus (D \cap D')$. Note that we can retrieve $F \times [0, 1]$ by cutting $V$ along $D$. Let $D^+, D^-$ be the copies of $D$ in $F \times \{1\}$, and let $\overline{\Delta}_1$ (resp. $\overline{\Delta}_2$) be the closure of $\Delta_1$ (resp. $\Delta_2$). Note that $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are disks properly embedded in $F \times [0, 1]$, and $\overline{\Delta}_i \cap (D^+ \cup D^-)$ consists of an arc properly embedded in $D^+ \cup D^-$. Let $\Gamma_i$ ($i = 1, 2$) be the disk in $F \times \{1\}$ such that $\partial \Gamma_i = \partial \overline{\Delta}_i$. Without loss of generality, we may suppose $\overline{\Delta}_1 \cap (D^+ \cup D^-) = \overline{\Delta}_1 \cap D^+$. Note that if $D^-$ is not contained in $\Gamma_1$, we can isotope $D'$ in $V$ via the product region between $\overline{\Delta}_1$ and $\Gamma_1$ to reduce $|D \cap D'|$, a contradiction. Hence, $D^-$ is contained in $\Gamma_1$. Let $\beta$ be the arc in $\partial D'$ as above. Then $\beta \cap D$ consists of finite number of points, say $p_0, p_1, \ldots, p_n$, where $\partial \beta = \{p_0, p_n\}$, $p_0 \in \partial \overline{\Delta}_1$, $p_n \in \partial \overline{\Delta}_2$, and $p_0, p_1, \ldots, p_n$ are arrayed on $\beta$ in this order. Then a small neighborhood of $p_0$ in $\beta$ is contained in a small neighborhood of $D^-$. If the other endpoint of the subarc $\overline{p_0p_1}$ of $\beta$ is contained in $\partial D^-$, then we see that the subarc $\overline{p_0p_1}$ is an inessential arc in $\text{Cl}(F \times \{1\} \setminus (D^+ \cup D^-))$. This shows that we can reduce $|D \cap D'|$ by an isotopy on $D'$, a contradiction. By applying the same
argument successively, we see that each subarc $p_{i-1}p_i$ ($i = 1, 2, \ldots, n$) joins $D^+$ and $D^-$, and particularly, a small neighborhood of $p_n$ in $\beta$ is contained in a small neighborhood of $D^+$. This shows that $\overline{\Delta_2} \cap (D^+ \cup D^-) = \overline{\Delta_2} \cap D^-$. Then we see that $D^+$ is not contained in $\Gamma_2$, hence we have a contradiction by using the argument as above. □

**Lemma 3.2.4.** Any essential separating disk in $V$ can be isotoped to be disjoint from the non-separating disk.

*Proof.* Let $D$ be an essential separating disk in $V$. Then $D$ cuts $V$ into a solid torus $T$ and a manifold homeomorphic to $F \times [0, 1]$, and there is a non-separating disk $D'$ in $T$ such that $D'$ is disjoint from $D$ in $V$. By Lemma 3.2.3, we see that $D'$ is unique. Hence we obtain the desired result. □

### 3.2.2 Heegaard splitting with distance exactly $n$

*Proof of Theorem 1.1.4.* Let $C_1$ and $C_2$ be copies of the compression body obtained by adding a 1-handle to $F \times [0, 1]$, where $F$ is a genus-$(g - 1)$ closed orientable surface ($g > 1$). Let $\alpha_0$ be the boundary of the meridian disk $D_1$ of $C_1$ and $\alpha_2$ a simple closed curve on $\partial_+ C_1$ which intersects $\alpha_0$ transversely in one point. Then we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ on $\partial_+ C_1$ as in Section 3.1.2. Note that $\alpha_{n+2}$ intersects $\alpha_n$ transversely in one point by the construction. Glue $\partial_+ C_1$ and $\partial_+ C_2$ by a homeomorphism that sends the boundary of an essential non-separating disk $D_2$ of $C_2$ to the curve $\alpha_{n+2}$. We denote the images of $\partial_+ C_1$ and $\partial_+ C_2$ by $P$. Then $C_1 \cup_P C_2$ is a genus-$g$ Heegaard splitting of a compact orientable 3-manifold.

Let $D_1'$ be an essential separating disk in $C_1$ disjoint from $\alpha_2$ obtained as follows. First we take two disks parallel to $D_1$. Connecting them by a band along $\alpha_2$ on $P$ produces the desired disk. Similarly, we can obtain an essential separating disk $D_2'$ in $C_2$ disjoint from $\alpha_n$. On the other hand, we have $d_P(\alpha_2, \alpha_n) = n - 2$ since $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ is a geodesic in $C(S)$. Hence,

$$d_P(\partial D_1', \partial D_2') \leq d_P(\partial D_1', \alpha_2) + d_P(\alpha_2, \alpha_n) + d_P(\alpha_n, \partial D_2')$$

$$= 1 + (n - 2) + 1$$

$$= n.$$
Let $D_1'' \subset C_1$ and $D_2'' \subset C_2$ be any essential disks. By Section 3.2.1, we have $d_P(\partial D_i'', \partial D_i) \leq 1$ for $i = 1, 2$. This implies

$$d_P(\partial D_1, \partial D_2) \leq d_P(\partial D_1, \partial D_1'') + d_P(\partial D_1'', \partial D_2'') + d_P(\partial D_2'', \partial D_2)$$

$$\leq 1 + d_P(\partial D_1'', \partial D_2'') + 1,$$

and hence

$$d_P(\partial D_1'', \partial D_2'') \geq d_P(\partial D_1, \partial D_2) - 2$$

$$= d_P(\alpha_0, \alpha_{n+2}) - 2$$

$$= (n + 2) - 2$$

$$= n.$$

Hence $d_P(\partial D_1'', \partial D_2'') \geq n$ for any pair of essential disks $D_1'' \subset C_1$ and $D_2'' \subset C_2$, which implies $d_P(D(C_1), D(C_2)) \geq n$. Since $d_P(\partial D_1', \partial D_2') \leq n$, this gives $d_P(D(C_1), D(C_2)) = n$. \hfill \Box

**Proof of Theorem 1.4.** Let $C_1 \cup_P C_2$ be a genus-$g$ Heegaard splitting with distance exactly $n$ as above. Let $K_g$ be the constant as in [25, Theorem 1.1]. Assume that $n \geq K_g + 2$. By applying [22, Theorem 5.3], we obtain a genus-$g$ Heegaard splitting $C_1 \cup_P C_2^*$ with distance exactly $n$, where $C_2^*$ is a handlebody. Then by applying [22, Theorem 1.3] to $C_1 \cup_P C_2^*$, we obtain a genus-$g$ Heegaard splitting $C_1^* \cup_P C_2^*$ with distance exactly $n$, where $C_1^*$ and $C_2^*$ are handlebodies. \hfill \Box

### 3.2.3 (1, 1)-splitting with distance exactly $n$

We apply the idea for the proof of Theorem 1.1.4 to genus-1 1-bridge knots. A knot $K$ in an orientable closed 3-manifold $M$ is called a genus-1 1-bridge knot, or a (1, 1)-knot in brief, if $(M, K) = (V_1, t_1) \cup_P (V_2, t_2)$, where $V_1 \cup_P V_2$ is a genus-1 Heegaard splitting and $t_i$ is a trivial arc in $V_i$ ($i = 1, 2$). Then $(V_1, t_1) \cup_P (V_2, t_2)$ is called a (1, 1)-splitting of $(M, K)$. For $i = 1$ or 2, $D(V_i)$ denotes the subset of $\mathcal{C}(P)$ consisting of the vertices with representatives bounding disks in $V_i - t_i$. Then the distance of $(V_1, t_1) \cup_P (V_2, t_2)$ is defined by $d_P(D(V_1), D(V_2))$. (For details, see [27].)
Theorem 3.2.5. For any even integer \( n > 0 \), there exists a \((1,1)\)-splitting with distance exactly \( n \).

Proof. Basically we mimic the proof of Theorem 1.1.4. The key fact is the following assertion proved by Saito [27, Proposition 3.8] which shows that \( \mathcal{D}(V_i \setminus t_i) \) has the same structure as \( \mathcal{D}(C_i) \) in the proof of Theorem 1.1.4 (\( i = 1, 2 \)).

Assertion 3.2.6. Let \( D_i \) be an essential disk in \( V_i \setminus t_i \) as in Figure 3.4. Then any non-separating essential disk in \( V_i \setminus t_i \) is isotopic to \( D_i \) and any separating essential disk in \( V_i \setminus t_i \) can be isotoped to be disjoint from \( D_i \).

\[ \text{Figure 3.4:} \]

Then starting with simple closed curves \( \alpha_0 = \partial D_1, \alpha_1 \) and \( \alpha_2 \) as in Figure 3.5, we apply the construction of a geodesic in Subsection 3.1.2. Then the argument in the proof of Theorem 1.1.4 enables us to show the existence of a \((1,1)\)-splitting with distance \( n \) for each even \( n \). \( \Box \)

\[ \text{Figure 3.5:} \]

Here we note that \( \alpha_1 \) is separating in \( \partial V_1 \setminus t_1 \). Hence, we cannot apply the extension of the geodesic described in Subsection 3.1.2.
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