Continuous Attractor that Appears in Autoassociative Memory Model Extended to XY Spin System

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We analyzed the attractor structure in an autoassociative memory model extended to an XY spin system. We examined the attractor structures for two kinds of memory patterns: one composed of two states, 0 and \(\pi\); and the other composed of an infinite number of states \([0, 2\pi]\). To focus on analyzing the attractor structure, we used a finite number of memory patterns. As a result of our study, we found that memory patterns composed of two states are acquired not as point attractors but as a continuous (line) attractor, and that the retrieval state of the model can change without the free energy changing from one memory pattern to another. The memory patterns are acquired as point attractors in an autoassociative memory model of an Ising spin system. Therefore, the attractor structure is changed markedly by extending the autoassociative memory model to an XY spin system. Furthermore, we found that the attractor structure changes depending on the number of states in the memory pattern, and that the memory patterns composed of an infinite number of states are not acquired as a continuous attractor.

KEYWORDS: XY spin, Hopfield model, point attractor, continuous (line) attractor, external input, statistical mechanics

1. Introduction

The autoassociative memory model of an Ising spin system proposed by Hopfield and its improved versions proposed by various researchers were examined to gain a deeper understanding of their storage capacity¹⁻³ and attractor structure.⁴⁻⁷ Studies of the attractor structure revealed that the model acquires memory patterns and symmetric mixed states that are mixed symmetrically with the memory patterns, as point attractors. Studies that extended the autoassociative memory model of an Ising spin system to an XY spin system did not examine the attractor structure in detail, although the storage capacity was examined.⁸⁻¹⁰

In this study, we focused on the attractor structure acquired by the autoassociative memory model of an XY spin system. We examined the difference in attractor structure for two kinds of memory patterns: one composed of two states, 0 and \(\pi\); and the other composed of

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an infinite number of states \([0, 2\pi]\). A finite number of memory patterns \(p = 3, 4, 5\) were used in order to focus on the attractor structure. Statistical mechanics and computer simulation were used for our examination.

As a result of our study, we found that, in the autoassociative memory model of an XY spin system, memory patterns composed of two states, 0 and \(\pi\), are not acquired independently as point attractors but are acquired as a continuous (line) attractor. In the model with a continuous attractor, the retrieval state can change without the free energy changing from one memory pattern to another. We also found that memory patterns composed of an infinite number of states \([0, 2\pi]\) are acquired not as a continuous attractor but as a structure combining point attractors and a continuous attractor. Moreover, although the symmetric mixed state is stable in the Ising spin system\(^4,\!)\(7\), it becomes unstable in the autoassociative memory model extended to an XY spin system.

A model storing memory patterns as point attractors must clear a high free energy barrier when the retrieval state changes from one memory pattern to another memory pattern. Achieving this requires putting strong stimuli in the external input so that the retrieval state changes from one memory pattern to another memory pattern. The retrieval state can easily be changed by adding an external input to a model with a continuous attractor. The results of our study revealed that the retrieval state changes even for a very small external input in the model of an XY spin system with a continuous attractor. If such a model has a finite temperature and a finite spin size, its retrieval state shows a random walk pattern on various states of the continuous attractor owing to the finite size effect. The random walk can be stopped and the memory pattern can be retained by adding an external input. This continuous attractor network model is interesting and important in that the retrieval state can easily be changed by adding an external input.

2. Model

The XY spin is a simplified spin model commonly used in statistical mechanics. The spin is a unit vector with the phase \(\phi\), which is a state variable; \(\phi\) is in the range of \(0 \leq \phi < 2\pi\). As shown in Fig. 1, the model consists of \(N\) spins \(X_i, 1 \leq i \leq N\), that have the phases \(\phi_i, 1 \leq i \leq N\) as state variables. The \(N\) spins have the mutual interaction \(J_{ij}\) from the \(j\)th spin to the \(i\)th spin and have the external input \(Y_i^{ext}\) through the positive coefficient \(h\). The external input \(Y_i^{ext}, 1 \leq i \leq N\) is a unit vector with the phase \(\psi_i^{ext}\) in the range of \(0 \leq \psi_i^{ext} < 2\pi\). The Hamiltonian \(H\) for this model is given by

\[
H = -\sum_{i<j}^N J_{ij}(X_i, X_j) - \sum_{i=1}^N h(X_i, Y_i^{ext}) \\
= -\sum_{i<j}^N J_{ij} \cos(\phi_i - \phi_j) - \sum_{i=1}^N h \cos(\phi_i - \psi_i^{ext}). \tag{1}
\]
When $J_{ij}$ is positive, the phases $\phi_i$ and $\phi_j$ will be the same owing to the effect of the first term on the right-hand side of Eq. (1). When $J_{ij}$ is negative, the phases $\phi_i$ and $\phi_j$ will be of antiphase to each other. If $h$ is sufficiently large, the phase $\phi_i$ becomes the same as the external input $\psi_{i}^{ext}$ owing to the effect of the second term on the right-hand side of Eq. (1).

The memory patterns $\theta_{\mu}^{i}, 1 \leq \mu \leq p$ stored in the model of an XY spin system are $N$-dimensional vectors consisting of $\theta_{\mu}^{i}, 1 \leq i \leq N$; $\mu$ is the serial number of a memory pattern. As mentioned earlier, we analyzed the attractor structure in an autoassociative memory model extended to the XY spin system for two kinds of memory patterns. In the “two-state memory pattern”, each element $\theta_{\mu}^{i}$ of the memory pattern $\theta_{\mu}^{i}$ has two states, 0 and $\pi$, and is generated by a discrete probability function:

$$\text{Prob}[\theta_{\mu}^{i} = 0] = \text{Prob}[\theta_{\mu}^{i} = \pi] = \frac{1}{2}, \quad 1 \leq i \leq N, 1 \leq \mu \leq p.$$  \hspace{1cm} (2)

In the “infinite-state memory pattern”, each element $\theta_{\mu}^{i}$ has continuous values in the range of $0 \leq \theta_{\mu}^{i} < 2\pi$ and is generated by a uniform probability function:

$$\text{Prob}[\theta_{\mu}^{i}] = \frac{1}{2\pi}, \quad 1 \leq i \leq N, 1 \leq \mu \leq p.$$  \hspace{1cm} (3)

The mutual interaction $J_{ij}$ is determined by a learning method:

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \cos(\theta_{\mu}^{i} - \theta_{\mu}^{j}), \quad 1 \leq i \leq N, 1 \leq j \leq N.$$  \hspace{1cm} (4)

In addition, we assume that $J_{ii} = 0$. Since $J_{ij}$ has a large positive value if the phase difference between $\theta_{\mu}^{i}$ and $\theta_{\mu}^{j}$ is small in accordance with the learning method, the phases $\phi_i$ and $\phi_j$ will be the same. Conversely, if the phase difference between $\theta_{\mu}^{i}$ and $\theta_{\mu}^{j}$ is large, the phases $\phi_i$ and $\phi_j$ will be antiphase to each other.

The order parameter $R_{\mu}$ indicates whether the phases $\phi_j, 1 \leq j \leq N$ correspond to the phase relation shown by $\theta_{\mu}^{j}, 1 \leq j \leq N$:

$$R_{\mu} = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i(\phi_{j} - \theta_{j}^{\mu})} \right|$$

$$= \left| \frac{1}{N} \sum_{j=1}^{N} \cos \phi_{j} \cos \theta_{j}^{\mu} + \frac{1}{N} \sum_{j=1}^{N} \sin \phi_{j} \sin \theta_{j}^{\mu} - i \frac{1}{N} \sum_{j=1}^{N} \cos \phi_{j} \sin \theta_{j}^{\mu} + i \frac{1}{N} \sum_{j=1}^{N} \sin \phi_{j} \cos \theta_{j}^{\mu} \right|$$

$$= \sqrt{(R_{\mu RR} + R_{\mu II})^2 + (R_{\mu RI} - R_{\mu IR})^2}, \quad 1 \leq \mu \leq p,$$  \hspace{1cm} (6)

where,

$$R_{\mu RR} = \frac{1}{N} \sum_{j=1}^{N} \cos \phi_{j} \cos \theta_{j}^{\mu}, \quad 1 \leq \mu \leq p,$$  \hspace{1cm} (7)

$$R_{\mu II} = \frac{1}{N} \sum_{j=1}^{N} \sin \phi_{j} \sin \theta_{j}^{\mu}, \quad 1 \leq \mu \leq p,$$  \hspace{1cm} (8)
\[ R_{\mu R} = \frac{1}{N} \sum_{j=1}^{N} \cos \phi_j \sin \theta_j^\mu, \quad 1 \leq \mu \leq p, \quad (9) \]

\[ R_{\mu I} = \frac{1}{N} \sum_{j=1}^{N} \sin \phi_j \cos \theta_j^\mu, \quad 1 \leq \mu \leq p. \quad (10) \]

If \( R_{\mu} = 1 \), the phases \( \phi_j, 1 \leq j \leq N \) correspond exactly to the phase relation shown by \( \theta_j^\mu, 1 \leq j \leq N \); if \( R_{\mu} = 0 \), the phases \( \phi_j, 1 \leq j \leq N \) and \( \theta_j^\mu, 1 \leq j \leq N \) are uncorrelated mutually.

When there is no external input, the absolute phases of the spins are undetermined because the Hamiltonian in this model is determined by eq. (1), and the value of the Hamiltonian does not change even if all spins are given the same phase rotation. Therefore, we define the equation for \( R_{\mu} \) so that \( R_{\mu} \) is determined by the relative phases. When there is an external input, this model determines the absolute phases of the spins.

### 3. Theory

In the theory we used for the calculations, the number of spins \( N \) is infinitely large and the number of memory patterns \( p \) is \( O(1) \) irrespective of \( N \). We analyzed the stable solution of this model at a finite temperature \( T \) using statistical mechanics. The Hamiltonian \( H \) is described by Eq. (11), which is based on eqs. (1) and (4). The external input \( \psi^{ext} \) can be a memory pattern or an unstored random pattern. The partition function \( Z \) is described by Eq. (12), and the free energy \( f \) for one spin is described by Eq. (13):

\[ H = -\sum_{i<j} N J_{ij} \cos(\phi_i - \phi_j) - \sum_{i=1}^{N} h \cos(\phi_i - \psi^{ext}_i) \]

\[ = -\frac{1}{2N} \sum_{\mu=1}^{p} \left[ \left( \sum_{i=1}^{N} \cos \phi_i \cos \theta_i^\mu \right)^2 + \left( \sum_{i=1}^{N} \sin \phi_i \sin \theta_i^\mu \right)^2 + \left( \sum_{i=1}^{N} \cos \phi_i \sin \theta_i^\mu \right)^2 + \left( \sum_{i=1}^{N} \sin \phi_i \cos \theta_i^\mu \right)^2 \right] \]

\[ -h \left( \sum_{i=1}^{N} \cos \phi_i \cos \psi^{ext}_i + \sum_{i=1}^{N} \sin \phi_i \sin \psi^{ext}_i \right). \quad (11) \]

\[ Z = \left( \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\phi_i \right) \exp(-\beta H), \quad (12) \]

\[ f = -\frac{1}{\beta N} \log Z, \quad (13) \]

where \( \beta = 1/T \). The free energy \( f \) is described by the order parameters given by Eqs. (7)-(10) and expressed as follows in terms of the thermodynamic limit of \( N \to \infty \):

\[ f = \frac{1}{2} \sum_{\mu=1}^{p} (R_{\mu RR}^2 + R_{\mu I I}^2 + R_{\mu RI}^2 + R_{\mu IR}^2) - \frac{1}{\beta} \ll \log \int_{-\pi}^{\pi} \exp(\Lambda) d\phi \gg \theta, \psi, \quad (14) \]

\[ \Lambda = \beta \sum_{\mu=1}^{p} (R_{\mu RR} \cos \phi \cos \theta^\mu + R_{\mu I I} \sin \phi \sin \theta^\mu + R_{\mu RI} \cos \phi \sin \theta^\mu + R_{\mu IR} \sin \phi \cos \theta^\mu) \]
\[ +\beta h \cos(\phi - \psi^{ext}), \]  

where \( \ll \cdot \gg_{\theta, \psi} \) stands for the average over the production probability of \( \theta^{\mu} \), as given by Eq. (2) or (3) and the production probability of \( \psi^{ext} \). The following order parameter equations are obtained using the saddle point conditions \( \frac{\partial f}{\partial R_{\mu RR}} = 0, \frac{\partial f}{\partial R_{\mu II}} = 0, \frac{\partial f}{\partial R_{\mu RI}} = 0, \) and \( \frac{\partial f}{\partial R_{\mu LR}} = 0 \), where \( f \) is minimized:

\[
R_{\mu RR} = \frac{\int_{-\pi}^{\pi} \cos \phi \cos \theta^\mu \exp(\Lambda) d\phi}{\int_{-\pi}^{\pi} \exp(\Lambda) d\phi}; \quad 1 \leq \mu \leq p, \quad \text{for } \theta, \psi \tag{16}
\]

\[
R_{\mu II} = \frac{\int_{-\pi}^{\pi} \sin \phi \sin \theta^\mu \exp(\Lambda) d\phi}{\int_{-\pi}^{\pi} \exp(\Lambda) d\phi}; \quad 1 \leq \mu \leq p, \quad \text{for } \theta, \psi \tag{17}
\]

\[
R_{\mu RI} = \frac{\int_{-\pi}^{\pi} \cos \phi \sin \theta^\mu \exp(\Lambda) d\phi}{\int_{-\pi}^{\pi} \exp(\Lambda) d\phi}; \quad 1 \leq \mu \leq p, \quad \text{for } \theta, \psi \tag{18}
\]

\[
R_{\mu LR} = \frac{\int_{-\pi}^{\pi} \sin \phi \cos \theta^\mu \exp(\Lambda) d\phi}{\int_{-\pi}^{\pi} \exp(\Lambda) d\phi}; \quad 1 \leq \mu \leq p, \quad \text{for } \theta, \psi \tag{19}
\]

\[
R_{\mu} = \sqrt{(R_{\mu RR} + R_{\mu II})^2 + (R_{\mu RI} - R_{\mu LR})^2}, \quad 1 \leq \mu \leq p. \tag{20}
\]

The values of \( R_{\mu RR}, R_{\mu II}, R_{\mu RI}, \) and \( R_{\mu LR} \) in the equilibrium state can be obtained by solving the simultaneous equations Eqs. (16)-(19) for various values of \( \beta \) and \( h \). If the signs of all eigenvalues of the Hessian matrix to be derived by differentiating \( f \) of Eq. (14) twice with respect to \( R_{\mu RR}, R_{\mu II}, R_{\mu RI}, \) and \( R_{\mu LR} \) are positive, the equilibrium state is stable. If one or more eigenvalues are zero, the equilibrium state has neutral stability.

4. Results

We first present the results of our examination of the two-state memory pattern in which the analysis was performed using numerical calculations based on Eqs. (16)-(19) and computer simulation as needed.

Figure 2 shows the temperature dependence of the stable solution for \( p = 3 \) and \( h = 0 \). In Fig. 2(a), the stable solution is the exact memory pattern \( \theta^1 \) at \( T = 0 \) since \( R_1 = 1 \) and \( R_2 = R_3 = 0 \). As \( T \) increases, \( R_1 \) gradually decreases because of state fluctuation due to temperature noise; at \( T = 0.5 \), \( R_1 = R_2 = R_3 = 0 \). In Fig. 2(b), the stable solution is the state in which \( \theta^1 \) and \( \theta^2 \) are mixed because \( R_1 < 1, R_2 > 0, \) and \( R_3 = 0 \) at \( T = 0 \). We call this state the “asymmetric mixed state”. As \( T \) increases, \( R_1 \) and \( R_2 \) of the asymmetric mixed state gradually decrease; at \( T = 0.5 \), \( R_1 = R_2 = R_3 = 0 \). Figure 2(c) shows the asymmetric mixed state in which the mixture ratio of \( \theta^1 \) to \( \theta^2 \) is different from that in the state shown in Fig. 2(b). In contrast, the symmetric mixed state of \( \theta^1, \theta^2, \) and \( \theta^3 \) does not become a stable solution. There have been no reports that the asymmetric mixed state becomes a stable solution in the Hopfield model of an Ising spin system. On the other hand, it has been reported that the symmetric mixed state becomes a stable solution in the Hopfield model of an Ising spin...
We found that the properties of the solution in the model of an XY spin system differ greatly from those of an Ising spin system. In Figs. 2(a)-2(c), the mixture ratios of $\theta^1$ to $\theta^2$ are mutually different. Furthermore, in addition to these asymmetric mixed states, there are many asymmetric mixed states with different mixture ratios. For any asymmetric mixed state, $R_1$ and $R_2$ gradually decrease when $T$ increases, and, at $T = 0.5$, $R_1 = R_2 = R_3 = 0$. Moreover, the same solutions were obtained for $p = 4$ and 5 as well.

Figure 3 shows the stable solutions in the model at $T = 0.1$ and $h = 0$. The stable solution on the left side of the graph shows the memory pattern $\theta^1$, the stable solution on the right side shows the memory pattern $\theta^2$, and the stable solutions between the two sides show the asymmetric mixed states for $\theta^1$ and $\theta^2$. The horizontal axis is set to be $\tan^{-1}(R_2/R_1)$ in order to show the point at which the stable solution changes from $\theta^1$ to $\theta^2$. It is understood that there are stable solutions that continuously change from $\theta^1$ to $\theta^2$. The symbols (filled circles, open circles, squares) in Figs. 2(a)-2(c) and the same symbols in Fig. 3 show the same solutions. Moreover, Fig. 3 shows that the free energy is invariant in the memory patterns $\theta^1$ and $\theta^2$, and all the asymmetric mixed states between $\theta^1$ and $\theta^2$. Consequently, it can be understood that there is a continuous attractor between $\theta^1$ and $\theta^2$, and that each stable solution of the model is one stable point on a continuous attractor. There are also continuous attractors in $\theta^1 \leftrightarrow \theta^3$ and $\theta^2 \leftrightarrow \theta^3$ because $\theta^1$, $\theta^2$, and $\theta^3$ have a symmetrical structure. The dashed line in Fig. 3 represents $\sqrt{R_1^2 + R_2^2 + R_3^2}$, which is constant at 0.95 for any stable solution. The same solutions were also obtained for $p = 4$ and 5.

In addition, we examined the eigenvalues of the Hessian matrix of the free energy for all memory patterns and all asymmetric mixed states in Fig. 3. The number of eigenvalues is $4p$ because the order parameters that describe the free energy are $R_{\mu RR}, R_{\mu RI}, R_{\mu RI}, R_{\mu RR}, 1 \leq \mu \leq p$. We found that three eigenvalues are zero in the stable solution for each of the memory patterns $\theta^1$, $\theta^2$, and $\theta^3$. One zeroeigenvalue means that the free energy is invariant even if the same amount of phase rotation occurs for all spins. The other two zeroeigenvalues mean that the free energy is invariant even if the retrieval state for a memory pattern changes toward either direction of another two memory patterns. Moreover, we found that two eigenvalues are zero for all the asymmetric mixed states. One zeroeigenvalue means that the free energy is invariant even if the same amount of phase rotation occurs for all the spins. The other zeroeigenvalue means that all the asymmetric mixed states have an invariant free energy between the two side memory patterns. These results also show that there is a continuous attractor mutually connecting all the memory patterns.

Figure 4 shows the change in the stable solution with a change in the strength of the external input for $p = 3$ and $T = 0.1$. In our experimental method of changing the stable solution, $\theta^1$ was first set as the initial state of the model. The filled circle on the upper left of Fig. 4 represents the stable solution for $\theta^1$ at $T = 0.1$. Next, $\theta^2$ was set as the external
input with various values for the coefficient $h$. The solid line in the figure shows the change in the stable solution with a change in the strength of the external input $h$. The stable solution rapidly changes to a state with $R_1 = 0$ when $h$ has a nonzero value. The solution for $R_1 = 0$ shows that $\theta^1$ becomes unstable and the stable solution changes to $\theta^2$. Therefore, the stable solution can change from $\theta^1$ to $\theta^2$ even if $h$ has a very small value and is not zero. The critical external input coefficient for $\theta^1$ to change to $\theta^2$ is $h_c = 0$. Also, this result means that the stable solutions of this model have a continuous attractor. The dashed curve shows the change in the stable solution with a change in the external input of an unstored random pattern. The stable solution gradually changes to a state with $R_1 = 0$ as $h$ increases. The solution for $R_1 = 0$ shows that $\theta^1$ becomes unstable and that the stable solution changes to an unstored random pattern. The critical external input coefficient for $\theta^1$ to change to an unstored random pattern is $h_c = 0.95$. Therefore, a very large $h$ is needed for an unstored random pattern to become a stable solution. The same solutions were also obtained for $p = 4$ and 5.

Figure 5 shows the critical external input coefficient $h_c$ at which the system state changes from the initial stable state for $\theta^1$ to the external input for various values of $T$. The critical external input coefficient at which the stable solution changes from $\theta^1$ to $\theta^2$ is $h_c = 0$ for $T < 0.5$. Therefore, the structure of the continuous attractor is maintained regardless of the temperature for $T < 0.5$. The critical external input coefficient at which the stable solution changes from $\theta^1$ to the unstored random pattern is $h_c = 1$ at $T = 0$, and $h_c$ gradually decreases as $T$ increases. The same solutions were also obtained for $p = 4$ and 5.

The symbols in Fig. 5 show the results of computer simulation ($N = 5,000; h_c$ examined every 0.005 steps). The computer simulation was executed in accordance with the Metropolis method, as described below. At time $t$, the phase of the $i$th spin chosen at random was assumed to be $\phi^i_t$. In accordance with the difference between the Hamiltonians $\Delta H = H(\phi^i_t + \Delta \phi_i) - H(\phi^i_t)$, the phase $\phi^{i+\Delta t}_t$ after $\Delta t$ is determined as follows:

\[
\begin{align*}
\text{when } \Delta H < 0 : \quad & \text{Prob}[\phi^{i+\Delta t}_t = \phi^i_t + \Delta \phi] = 1 \\
\text{else} : \quad & \text{Prob}[\phi^{i+\Delta t}_t = \phi^i_t + \Delta \phi] \\
& = 1 - \text{Prob}[\phi^{i+\Delta t}_t = \phi^i_t] = \exp^{-\beta \Delta H},
\end{align*}
\]

where $\Delta \phi_i$ is randomly in the range of $-\pi \leq \Delta \phi_i < +\pi$. Choosing a spin $N$ times and performing the update procedure is considered to be one Monte Carlo step (MCS). The computer simulation was executed for 10,000 MCSs so that the system state could reach equilibrium. The results are consistent with those of numerical calculation based on the theory.

These results show that the attractor structure when the model of an XY spin system stores two-state memory patterns with no external input is that shown in Fig. 6. The attractor of the model has a gutter-shaped structure, and its memory patterns are located at the bottom
of the gutter. The three balls represent the memory patterns $\theta^1$, $\theta^2$, and $\theta^3$. The bottom of the gutter connecting the memory patterns is flat. The gutter represents a continuous attractor. There is no gutter leading to $\theta^3$ in the middle state between $\theta^1$ and $\theta^2$. Since there is a flat gutter between the memory patterns, the retrieval state of the model easily changes from a retrieved memory pattern to another memory pattern with a very small external input. An unstored random pattern is located away from the gutter and has high free energy, so that a very large external input is needed to retrieve the unstored random pattern. The attractor structure for $p \geq 4$ is not shown in the figure because the true structure image cannot be visualized; however, gutters connect all memory patterns to each other for $p \geq 4$ as well.

Since the free energy is invariant between the memory patterns, the retrieval state of the model has no attractive force to remain in one memory pattern if the model has no external input. We examined the retrieval dynamics in the model with no external input. Computer simulation was performed for the examination at $p = 3$, $h = 0$, $T = 0.2$, and $N = 10,000$, and the state was updated in accordance with the Metropolis method. The memory pattern $\theta^1$ was used as the initial state of the retrieval process. Figure 7 shows that the long-term trajectories of $R_1$, $R_2$, and $R_3$ in the retrieval process do not remain in the memory pattern $\theta^1$ even though the memory pattern is stable at $T = 0.2$. The long-term trajectories move randomly at the bottom of the gutter in Fig. 6. That is, the long-term trajectories move around the memory patterns and various asymmetric mixed states at random. As shown in Fig. 2, the retrieval state remains in a memory pattern or in an asymmetric mixed state because the theoretical results are derived for $N = \infty$. However, as shown in Fig.7, the long-term trajectories exhibit a random walk on the continuous attractor because the computer simulation was executed for a finite $N$. When $T$ is further increased, the long-term trajectories begin to move around more quickly. The random walk of long-term trajectories can be stopped by adding an external input.

We also investigated the short-term trajectory in the initial retrieval process. The computer simulation was executed for $N = 10,000$. The initial state was set to that for various overlaps with $\theta^1$. The short-term trajectory for 0-100 MCS in the retrieval process was examined for $p = 3$, $h = 0$, and $T = 0.2$. Figure 8 shows the short-term trajectories when the initial overlaps were set to $R_1$(time = 0) = 0.1, 0.2, $\cdots$, 1.0 and $R_2$(time = 0) = $R_3$(time = 0) = 0. The trajectories move toward the state of $R_1 = 0.86$ and $R_2 = R_3 = 0$ for time = 20 MCSs. This means that the short-term trajectories move toward the memory pattern $\theta^1$ because $R_1$, $R_2$, and $R_3$ are the same as those at $T = 0.2$ in Fig. 2(a). Subsequently, as shown in Fig. 7, the trajectories begin to move around various asymmetric mixed states at random after they reach the memory pattern $\theta^1$. Since the free energy space of the model has a gutter-shaped structure and the memory pattern is located at the bottom of the gutter, the trajectories apparently move toward the memory pattern in the initial retrieval process and exhibit a
random walk pattern around various asymmetric mixed states afterwards.

To compare the attractor structure with that in the model of an XY spin system storing the two-state memory patterns, we examined the attractor structure in the model of an XY spin system storing the infinite-state memory patterns. Figure 9 shows the temperature dependence of the stable solution for $p = 3$ and $h = 0$. The stable solution is the exact memory pattern $\theta^1$ at $T = 0$ since $R_1 = 1$ and $R_2 = R_3 = 0$. When $T$ increases, $R_1$ gradually decreases because of state fluctuation due to temperature noise at $T = 0.25$, $R_1 = R_2 = R_3 = 0$. In this model, the asymmetric mixed state is not a stable solution. Moreover, the symmetric mixed state of $\theta^1, \theta^2$, and $\theta^3$ is not a stable solution. The same solutions were also obtained for $p = 4$ and 5.

Figure 10 shows the change in the stable solution with a change in the strength of the external input $h$ for $T = 0.1$. The experimental method was the same as that used to obtain the results shown in Fig. 4. The solid curve shows the change for the external input $\theta^2$. The stable solution rapidly changes to the state with $R_1 = 0$ as $h$ increases. The solution of $R_1 = 0$ means that $\theta^1$ becomes unstable and that the stable solution changes to $\theta^2$. The critical external input coefficient necessary for $\theta^1$ to change to $\theta^2$ is $h_c = 0.07$. The $h_c$ for the external input $\theta^2$ is larger than that when the two-state memory patterns are stored. The dashed curve shows the change in the stable solution with a change in the external input of an unstored random pattern. The stable solution gradually changes to the state with $R_1 = 0$ as $h$ increases. The solution of $R_1 = 0$ means that $\theta^1$ becomes unstable and that the stable solution changes to the unstored random pattern. The critical external input coefficient necessary for $\theta^1$ to change to the unstored random pattern is $h_c = 0.27$. The $h_c$ for an unstored random pattern is smaller than that when the two-state memory patterns are stored. The same solutions were also obtained for $p = 4$ and 5.

Figure 11 shows the critical external input coefficient at which the system state changes from the initial stable state for $\theta^1$ to the external input pattern for various values of $T$. The critical external input coefficient at which the stable solution changes from $\theta^1$ to $\theta^2$ is $h_c = 0.11$ at $T = 0$, and $h_c$ gradually decreases as $T$ increases. The critical external input coefficient at which the stable solution changes from $\theta^1$ to the unstored random pattern is $h_c = 0.33$ at $T = 0$, and $h_c$ gradually decreases as $T$ increases. The same solutions were also obtained for $p = 4$ and 5.

The symbols in Fig. 11 show the results of the computer simulation ($N = 5,000$; $h_c$ examined every 0.005 steps). The computer simulation was executed in accordance with the Metropolis method for 10,000 MCSs so that the system could reach equilibrium. The results of computer simulation are consistent with those of numerical calculations based on the theory.

Let us imagine the attractor structure in the model of an XY spin system storing infinite-state memory patterns. In accordance with the results described above, the $h_c$ for the external input $\theta^2$ is larger than that when the two-state memory patterns are stored, and the $h_c$ for an
unstored random pattern is smaller than that when the two-state memory patterns are stored. Therefore, the attractor structure when the model of an XY spin system stores infinite-state memory patterns with no external input is that shown in Fig. 12. The attractor of the model has a gutter-shaped structure, and the memory patterns are located at the bottom of the gutter. This model has a structure similar to that of a continuous attractor, but the bottom of the gutter arches up between memory patterns. Therefore, the attractor structure is not a continuous attractor structure. Since the gutter arch has a high elevation and the model acquires memory patterns as point attractors, the retrieval state easily remains near a memory pattern even if the model has no external input and the temperature is high. A large external input is needed to change the retrieval state from one memory pattern to another. Moreover, the difference in free energy barrier when the retrieval state changes from one memory pattern to another and that when the retrieval state changes from one memory pattern to an unstored random pattern is smaller than that when the two-state memory patterns are stored. The attractor structure strongly depends on the number of states in the memory pattern.

Since the attractor structure greatly depends on the number of states in the memory pattern, we determined whether the attractor structure of the model changes when the number of memory pattern states is increased by adding a Gaussian random number to the two-state memory patterns. Figure 13 shows the probability density function of the two-state memory pattern $\theta^\mu$ with an added Gaussian random number with a standard deviation $\sigma = 0.3$. This memory pattern was used in a computer simulation because the memory pattern is unstable in this model of an XY spin system when $\sigma > 0.35$.

The results for $p = 3$, $N = 10,000$ and $h_c$ examined every 0.005 steps are plotted in Fig. 14, which shows the $h_c$ values for various $T$ values when the system state changes from the initial stable state for $\theta^1$ to the state for the external input. The stable solution changes from $\theta^1$ to $\theta^2$ at $h_c = 0$ for $T < 0.47$. Thus, the structure of the continuous attractor is maintained regardless of the temperature for $T < 0.47$. The critical external input coefficient at which the stable solution changes from $\theta^1$ to the unstored random pattern is $h_c = 0.98$ at $T = 0$, and $h_c$ gradually decreases as $T$ increases. The $h_c$ for the unstored random pattern is slightly smaller than that for the two-state memory pattern (Fig. 5) owing to the added Gaussian random number. Consequently, the structure of the continuous attractor is maintained even if the number of states in the memory patterns is increased by adding a Gaussian random number. It is thought that the maintenance of the continuous attractor structure requires that the probability density function of the memory pattern have two peaks.

5. Conclusions

In this study, we analyzed an attractor structure in an autoassociative memory model of an XY spin system using statistical mechanics and computer simulation. We examined the difference in attractor structure for two kinds of memory patterns: one composed of two
states, 0 and $\pi$; and the other composed of an infinite number of states $[0, 2\pi)$. We used a finite number of patterns $p = 3, 4, 5$ in order to focus on the attractor structure.

The results indicated that, in the autoassociative memory model in an XY spin system storing memory patterns composed of two states, 0 and $\pi$, the memory patterns are not independently acquired as point attractors but are acquired as a continuous (line) attractor, and that there are various asymmetric mixed states functioning as a continuous attractor between memory patterns. If the memory patterns are acquired as a continuous attractor, it is easy to change the retrieval state of the model from one memory pattern to another by adding an external input. A memory pattern is acquired as a point attractor in an autoassociative memory model of an Ising spin system. Therefore, the attractor structure is greatly changed by extending the autoassociative memory model to an XY spin system. The results also revealed that memory patterns composed of an infinite number of states $[0, 2\pi)$ are acquired not as a continuous attractor but as a structure combining point attractors and a continuous attractor. The attractor structure depends on the number of states in the memory pattern. This continuous attractor network model is interesting and important in that the retrieval state can easily be changed by adding an external input.

Analysis results for the phase transition temperature in which the memory pattern and asymmetric mixed state become unstable and for the continuous attractor in an autoassociative memory model of an XY spin system storing correlated memory patterns will be reported in the future.

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Fig. 1. Model of XY spin system
Fig. 2. Temperature dependences of various stable solutions in XY spin system (two-state memory pattern, \( p = 3 \), and \( h = 0 \)). Each curve represents theoretical results. Each symbol shows the same solutions as those in Fig. 3. The same solutions were also obtained for \( p = 4, 5 \).
Fig. 3. Stable solutions and free energy showing structure of continuous attractor (two-state memory pattern, $p = 3$, $h = 0$, and $T = 0.1$). The left side of the graph represents the memory pattern $\theta^1$, and the right side represents the memory pattern $\theta^2$. Various asymmetric mixed states exist continuously as stable solutions between $\theta^1$ and $\theta^2$. The free energy is constant between $\theta^1$ and $\theta^2$. The same solutions were also obtained for $p = 4, 5$.

Fig. 4. Change in stable solution with change in strength of external inputs. The initial stable state was $\theta^1$. The external inputs were $\theta^2$ and unstored random pattern (two-state memory pattern, $p = 3$, and $T = 0.1$). The solid line is the solution for the external input $\theta^2$, and the dashed curve is the solution for an unstored random pattern. The solutions of $R_1 = 0$ mean that the system state is $\theta^2$ or an unstored random pattern. The same solutions were also obtained for $p = 4, 5$. 

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Fig. 5. Critical external input coefficient at which system state changes from initial stable state for $\theta^1$ to external input pattern. The external inputs were $\theta^2$ and an unstored random pattern (two-state memory pattern, $p = 3$, and $T = 0.1$). The symbols are computer simulation results ($N = 5,000$). The same solutions were also obtained for $p = 4, 5$.

Fig. 6. Image of continuous attractor in XY spin system (two-state memory pattern, $p = 3$, and $h = 0$).
Fig. 7.  Long-term trajectories of $R_1$, $R_2$, and $R_3$ in retrieval process obtained by computer simulation (two-state memory pattern, $p = 3$, $h = 0$, $T = 0.2$, and $N = 10,000$).

Fig. 8.  Short-term trajectories of $R_1$, $R_2$, and $R_3$ in initial retrieval process obtained by computer simulation (two-state memory pattern, $p = 3$, $h = 0$, $T = 0.2$, and $N = 10,000$).
Fig. 9. Temperature dependence of various stable solutions in XY spin system (infinite-state memory pattern, \( p = 3 \), and \( h = 0 \)). The each line represents the theoretical results. The same solutions were also obtained for \( p = 4, 5 \). The asymmetric mixed state did not become a stable solution.

Fig. 10. Change in stable solution with change in strength of external inputs. The initial stable state was \( \theta^1 \). The external inputs were \( \theta^2 \) and an unstored random pattern (infinite-state memory pattern, \( p = 3 \), and \( T = 0.1 \)). The solid curve represents the solution for the external input \( \theta^2 \), and the dashed curve represents the solution for an unstored random pattern. The solutions of \( R_1 = 0 \) mean that the system state is \( \theta^2 \) or an unstored random pattern. The same solutions were also obtained for \( p = 4, 5 \).
Fig. 11. Critical external input coefficient at which state of system changes from initial stable state $\theta^1$ to external input pattern. The external inputs were $\theta^2$ and an unstored random pattern (infinite-state memory pattern; $p = 3$). The symbols are computer simulation results ($N = 5,000$). The same solutions were also obtained for $p = 4, 5$.

Fig. 12. Image of attractor in XY spin system (infinite-state memory pattern and $h = 0$).

Fig. 13. Probability density function of two-state memory pattern with Gaussian random number ($\sigma = 0.3$).
Fig. 14. Critical external input strength at which system state changes from initial stable state $\theta^1$ to external input pattern. The external inputs were $\theta^2$ and an unstored random pattern. The symbols are computer simulation results (two-state memory pattern, $\sigma = 0.3$, $p = 3$, and $N = 10,000$).
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