Analysis of a solvable model of a phase oscillator network on a circle with infinite-range Mexican-hat-type interaction

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We study a phase oscillator network on a circle with an infinite-range interaction. First, we treat the Mexican-hat interaction with the zeroth and first Fourier components. We give detailed derivations of the auxiliary equations for the phases and self-consistent equations for the amplitudes. We solve these equations and characterize the nontrivial solutions in terms of order parameters and the rotation number. Furthermore, we derive the boundaries of the bistable regions and study the bifurcation structures in detail. Expressions for location-dependent resultant frequencies and entrained phases are also derived. Secondly, we treat a different interaction that is composed of $m$th and $n$th Fourier components, where $m < n$, and we study its nontrivial solutions. In both cases, the results of numerical simulations agree quite well with the theoretical results.

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I. INTRODUCTION

Synchronization phenomena are ubiquitous in nature, and they are very important to living organisms. Typical examples include the simultaneous emission of light by fireflies, the rhythm of the heart composed of a population of cardiac muscle cells, and circadian rhythms [1,2].

Pioneering studies on such behavior were done by Winfree [3] and Kuramoto [4]. In particular, Kuramoto regarded synchronization as a phase transition and devised a model in which synchronization occurs as a phase transition in a nonequilibrium system. In general, when nonlinear dynamical systems with stable limit cycle oscillators are weakly coupled, the whole system can be described in terms of the phases of the oscillators, and the dynamical equation reduces to the evolution equation for phases [4]. Kuramoto proposed the so-called Kuramoto model, which is a coupled phase oscillator network. He found that the synchronization-desynchronization phase transition takes place when the system parameter reaches a certain value, and he derived an analytic expression for the critical point [5]. Since Kuramoto’s analysis of globally coupled oscillators, oscillator networks with both short-range and intermediate-range interactions have been studied [6]. Oscillators with global and random interactions [7] and with sparse and random interactions [8] have also been studied. Moreover, a number of studies have analyzed the stability of the stationary states [9–11]. Many of these studies use the Fokker-Planck equation to derive a phase distribution density function with or without external noise. In particular, Otto and Antonsen derived evolution equations for the order parameters by assuming a special form for the Fourier components of the phase distribution density function [12]. Since this study, many studies have been published on the dynamical behavior of order parameters [13,14]. One of the interesting findings is the so-called Chimera state, in which some fraction of the oscillators are perfectly synchronized while the remainder are desynchronized [15–17]. Another topic on general coupled oscillator networks is noise-induced synchronization. It has been found that two identical nonlinear oscillators synchronize in the presence of common external Gaussian noise [18]. The findings of this study have been extended to include systems with common and oscillator-dependent noise [19,20], noise in the form of random impulses [21], and common noise consisting of only two values [22]. A review of the Kuramoto model and its extensions is available elsewhere [23]. There have also been extensive studies on the statistical and dynamical properties of the mean-field $XY$ model (HMF $XY$ model) of conservative dynamical systems that are related to oscillator network models of dissipative dynamical systems [24,25].

In our previous study [26], we investigated global coupled phase oscillators arranged on a circle. The interaction between two elements depended on their distance. In particular, we studied the Mexican-hat interaction, which is used in neuroscience studies to model feature extraction cells in the visual cortex and embodies the property that a firing cell excites nearby cells and inhibits distant cells [27]. The interaction is composed of the zeroth and first Fourier components. We proposed a method to derive auxiliary equations that enable us to determine the phases of three complex order parameters completely. By using these phases, we obtained expressions for the self-consistent equations (SCEs) of the amplitude of the order parameters and equations for the boundaries of the bistable regions. We performed numerical simulations and found that they agreed quite well with the theoretical results.

In this study, first we treat the interaction with the zeroth and first Fourier components. We call the Mexican-hat interaction model 1. This model was studied previously. In this paper, we give a derivation of the auxiliary equations for the phases and self-consistent equations for the amplitudes, solve these equations, and derive the boundaries of the bistable regions. We obtain three nontrivial solutions that are characterized by the order parameters and the rotation numbers of the synchronized oscillators. We draw the phase diagram by using formulas for the phase boundaries derived using the unstable pendulum ($Pn$) solution. We find that the disappearance of coexistent regions between the $U$ and $S$ solutions and between the

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The structure of this paper is as follows. From Secs. II–VI, we study model 1. In Sec. II, we formulate the problem and describe the SCEs and auxiliary equations. The solutions of the auxiliary equations are also given. In Sec. III, we characterize the nontrivial solutions by introducing the rotation number. In Sec. IV, we give the SCEs for the nontrivial solutions. The phase diagram and bifurcation structure are studied in Sec. V. In Sec. VI, we show the results of numerical simulations and compare them with the theoretical results. In Sec. VII, we derive the SCEs and relevant quantities for each phase.

Let us consider N equally spaced phase oscillators lying on a circle. We introduce the coordinate \( \theta \) on the circle, which takes values \( 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{(N-1)\pi}{N} \). Let \( \phi_0 \) be the phase of the oscillator at the coordinate \( \theta \), and assume that it obeys the following differential equation:

\[
\frac{d}{dt} \phi_0 = \omega_0 + \sum_{\theta'} J_{0, \theta'} \sin(\phi_{\theta'} - \phi_0). \tag{1}
\]

Here, \( \omega_0 \) is the natural frequency, and it is drawn from a probability density \( g(\omega) \). \( g(\omega) \) is assumed to be one-humped at \( \omega = \omega_0 \) and symmetric with respect to \( \omega_0 \). The interaction \( J_{0, \theta'} \) between oscillators at \( \theta \) and \( \theta' \) is assumed to be

\[
J_{0, \theta'} = \frac{J_0}{N} + \frac{J_1}{N} \cos(\theta - \theta'). \tag{2}
\]

This interaction has the properties of the Mexican-hat interaction described above. Now, let us introduce three complex order parameters,

\[
W = Re^{i\Theta_0} = \frac{1}{N} \sum_{\theta} e^{i \phi_0}, \tag{3}
\]

\[
W_c = Re^{i\Theta} = \frac{1}{N} \sum_{\theta} \cos \theta e^{i \phi_0}, \tag{4}
\]

\[
W_s = Re^{i\Theta_s} = \frac{1}{N} \sum_{\theta} \sin \theta e^{i \phi_0}. \tag{5}
\]

By using these quantities, the evolution equation (1) can be rewritten as

\[
\frac{d}{dt} \phi_0 = \omega_0 + J_0 R \sin(\Theta - \phi_0) + J_1 [R_c \cos \theta \sin(\Theta_c - \phi_0) + R_s \sin \theta \sin(\Theta_s - \phi_0)]. \tag{6}
\]

Let us study the stationary states of this equation. Since oscillators with a natural frequency \( \omega_0 \) are the most numerous, it is expected that the phases of the order parameters rotate with the frequency \( \omega_0 \) in the stationary states. Thus, we will assume the following relations:

\[
\Theta = \omega_0 t + \Theta', \quad \Theta_c = \omega_0 t + \Theta_c', \quad \Theta_s = \omega_0 t + \Theta_s'.
\]

Since we are studying stationary states, we will assume that the amplitudes \( R, R_c, R_s \) and the phases \( \Theta', \Theta_c', \Theta_s' \) tend to constant values as \( t \) goes to infinity. We could assume \( \omega_0 = 0 \) without loss of generality. However, we will retain the term \( \omega_0 \) because the parameters at critical points explicitly contain \( \omega_0 \).

Now, let us derive the SCEs following Kuramoto’s argument. We rewrite the right-hand side of Eq. (6) as

\[
\frac{d}{dt} \phi_0 = \omega_0 - A_0 \sin(\phi_0 - \omega_0 t - \phi_0). \tag{7}
\]

From Eqs. (6) and (7), the following relation is derived:

\[
A_0 e^{i\alpha_0} = J_0 Re^{i\Theta} + J_1 [R_c \cos \theta e^{i\Theta_c} + R_s \sin \theta e^{i\Theta_s}]. \tag{8}
\]

For simplicity, we will omit primes from the phases except for the expressions of \( \alpha_0 \) and \( \phi_0 \). \( A_0 \) is expressed as

\[
A_0^2 = (J_0 R)^2 + J_1^2 [(R_c \cos \theta)^2 + (R_s \sin \theta)^2 + 2 J_0 J_1 R_c \cos(\Theta_c - \Theta_s) \sin \theta \cos \theta + 2 J_0 J_1 R_s \cos(\Theta_s - \Theta_c) \cos \theta - R_c \cos(\Theta_c - \Theta_s) \sin \theta]. \tag{9}
\]

Since we assume that \( \Theta \)'s and \( R \)'s do not depend on time, neither does \( \alpha_0 \). Thus, by defining \( \psi_0 \equiv \phi_0 - \omega_0 t - \phi_0 \), the evolution equation becomes

\[
\frac{d}{dt} \psi_0 = \omega_0 - \omega_0 - A_0 \sin \psi_0. \tag{10}
\]

A. Synchronized oscillators: \( |\omega_0 - \omega_0| \leq A_0 \)

The stable and unstable solutions are as follows:

\[
\begin{align*}
\text{Stable solutions:} & \quad 0 < \psi_0 < \frac{\pi}{2} \quad \text{for} \quad \omega_0 - \omega_0 > 0 \\
& \quad -\frac{\pi}{2} < \psi_0 < 0 \quad \text{for} \quad \omega_0 - \omega_0 > 0, \\
\text{Unstable solutions:} & \quad \frac{\pi}{2} < \psi_0 < \pi \quad \text{for} \quad \omega_0 - \omega_0 > 0 \\
& \quad -\pi < \psi_0 < -\frac{\pi}{2} \quad \text{for} \quad \omega_0 - \omega_0 > 0.
\end{align*}
\]

The entrained phase \( \psi_0^* \) and number density of the synchronized oscillators with phase \( \psi \) at the location \( \theta, n_s(\theta, \psi) \) are obtained as

\[
\psi_0^* = \sin^{-1} \left( \frac{\omega_0 - \omega_0}{A_0} \right), \tag{11}
\]

\[
n_s(\theta, \psi) = g(\omega_0 + A_0 \sin \psi) A_0 \cos \psi, \quad |\psi| \leq \frac{\pi}{2}. \tag{12}
\]
where $\sin^{-1}(x)$ is assumed to be the principal value and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

**B. Desynchronized oscillators: $|\omega_\theta - \omega_\delta| > A_\theta$**

The solution of Eq. (10) is

$$\psi_\theta(t) = \tilde{\omega}_\theta t + h(\tilde{\omega}_\theta t),$$

where $h(t)$ is a periodic function of $t$ with period $2\pi$. $\tilde{\omega}_\theta$ is the resultant frequency given by

$$\tilde{\omega}_\theta = (\omega_\theta - \omega_\delta) \sqrt{1 - \left(\frac{A_\theta}{\omega_\theta - \omega_\delta}\right)^2}.$$  

The solution of Eq. (7) is

$$\phi_\theta(t) = \omega_\theta t + \psi_\theta(t) + \alpha_\theta.$$  

Therefore, the resultant frequency $\tilde{\omega}_0$ for $\phi_\theta(t)$ is

$$\tilde{\omega}_0 = \omega_\theta + \omega_\delta = (\omega_\theta - \omega_\delta) \sqrt{1 - \left(\frac{A_\theta}{\omega_\theta - \omega_\delta}\right)^2}. \quad (13)$$

The probability density function of the phase of desynchronized oscillators at location $\theta$, $p_{\delta\theta}(\theta, \psi)$, obeys the following equation:

$$\frac{\partial}{\partial \psi}((\omega_\theta - \omega_\delta - A_\theta \sin(\psi)) p_{\delta\theta}(\theta, \psi)) = 0.$$  

Solving it yields

$$p_{\delta\theta}(\theta, \psi) = \frac{|\omega_\theta - \omega_\delta|}{2\pi} \frac{1}{|\omega_\theta - \omega_\delta - A_\theta \sin(\psi)|} \times \sqrt{1 - \left(\frac{A_\theta}{\omega_\theta - \omega_\delta}\right)^2}. \quad (14)$$

From this, the number density of desynchronized oscillators with phase $\psi$ at $\theta$, $n_{\delta\theta}(\theta, \psi)$, can be written as

$$n_{\delta\theta}(\theta, \psi) = \frac{1}{\pi} \int_{0}^{\infty} g(\omega) p_{\delta\theta}(\theta, \psi) d\omega \times x_g(\omega_\theta + x) \sqrt{x^2 - A_\delta^2 \sin^2(\psi)}. \quad (15)$$

**C. Resultant frequency distribution**

We study the resultant frequency distribution for the synchronized and desynchronized oscillators at $\theta$, $G_s(\omega, \theta)$ and $G_{\delta\theta}(\omega, \theta)$, $G_s(\omega, \theta)$ is

$$G_s(\omega, \theta) = \frac{N_{\omega, \theta}}{N} \delta(\omega - \omega_\theta), \quad (16)$$

where $N_{\omega, \theta}$ is the number of synchronized oscillators located in $(\theta, \theta + d\theta)$, and $\delta(x)$ is the Dirac delta. $G_{\delta\theta}(\omega, \theta)$ is expressed as

$$G_{\delta\theta}(\omega, \theta) = \frac{|\omega - \omega_0|}{\sqrt{(\omega - \omega_0)^2 + A_\delta^2}} g[\omega_\theta + \sqrt{(\omega - \omega_0)^2 + A_\delta^2}]. \quad (17)$$

The following relation is used in the derivation:

$$\omega_0 = \omega_\theta + (\tilde{\omega}_0 - \omega_\delta) \sqrt{1 + \left(\frac{A_\theta}{\omega_\theta - \omega_\delta}\right)^2}. \quad (18)$$

**D. SCEs, auxiliary equations, and solutions of auxiliary equations**

In this subsection, we state the SCEs, the auxiliary equations, and the solutions of the auxiliary equations. Their derivations are in Appendix A.

Here, we will introduce the following notation:

$$\langle B \rangle \equiv \frac{1}{\pi} \int_{0}^{\pi/2} d\psi \int_{0}^{2\pi} d\theta g(\omega_\theta + A_\theta \sin(\psi)) \cos^2(\theta) B,$$

$$Z \equiv \frac{1}{\pi} \int_{0}^{\pi/2} d\psi \int_{0}^{2\pi} d\theta g(\omega_\theta + A_\theta \sin(\psi)) \cos^2(\psi). \quad (19)$$

The SCEs are

$$R = (J_0 R + J_1 (R_1 \cos(\theta) \cos(\Theta_1 - \Theta) + R_1 \sin(\theta) \cos(\Theta_1 - \Theta))) Z,$$

$$R_1 = (J_0 (R_1 \cos(\theta) \cos(\Theta_1 - \Theta) + J_1 (R_1 \cos^2(\theta) \sin(\Theta_1 - \Theta) + R_1 \sin(\theta) \cos(\Theta_1 - \Theta))) Z, \quad (20)$$

The auxiliary equations are

$$R_1 \sin(\Theta_1 - \Theta) + R_1 (\sin(\theta) \sin(\Theta_1 - \Theta)) \sin(\Theta_1 - \Theta) = 0, \quad (24)$$

$$J_0 R_1 (\sin(\theta) \sin(\Theta_1 - \Theta)) \sin(\Theta_1 - \Theta) = 0, \quad (25)$$

Two of the auxiliary equations are independent. Thus, the phases of the complex order parameters are completely determined from the auxiliary equations. We list their solutions below.

(i) $R = 0, R_1 \neq 0. \quad (22)$

(ii) $R \neq 0, R_1 \neq 0. \quad (23)$

A solution of $\sin(\Theta_1 - \Theta) = 0$ and $\cos(\Theta_1 - \Theta) = 0$, that is, $\Theta_1 - \Theta = \pm \frac{\pi}{2} \ (\text{mod} \ 2\pi)$. This corresponds to the stable spinning (S) solution.

B solution of $\sin(\Theta_1 - \Theta) = 0$ and $\cos(\Theta_1 - \Theta) = 0$, that is, $\Theta_1 - \Theta = \Theta_1 - \Theta = 0$ or $\pi \ (\text{mod} \ 2\pi)$. This corresponds to the unstable $P\pi$ solution.

Hereafter, we omit “(mod 2\pi)” for simplicity.

**III. CHARACTERIZATION OF THE SOLUTIONS**

The preceding section determined the phases of the order parameters from the auxiliary equations. Furthermore,
Appendix B derives four solutions of the SCEs. These solutions are classified on the basis of the values of $R$ and $R_1$:

- **P**: paramagnetic solution, $(R, R_1) = (0, 0)$,
- **U**: uniform solution, $(R, R_1) = (+, 0)$,
- **S**: spinning solution, $(R, R_1) = (0, +)$, $(\Theta_e - \Theta_s) = \pm \frac{\pi}{2}$.
- **$Pn$**: pendulum solution, $(R, R_1) = (+, +)$,

\[
\{\Theta_e - \Theta_s, \Theta_e - \Theta_e\} = \left\{\pm \frac{\pi}{2}, \pm \frac{\pi}{2}\right\} : \text{stable},
\]

\[
\{\Theta_e - \Theta_s, \Theta_e - \Theta_e\} = \{0 \text{ or } \pi, \pm \frac{\pi}{2}\} : \text{unstable}.
\]

Let us study the physical meanings of these solutions. To characterize them, we will define the rotation number of a synchronized oscillator $X_{\phi^*}$, which depends on $\omega_i$, $\alpha_0$, and $\Omega$. In all solutions, the behavior of synchronized oscillators with entrained phases $\phi_{\Omega}^* = \omega_i + \psi_0 + \alpha_0$. The rotation number is the number of rotations of a synchronized oscillator $X_{\phi^*} = (\cos \phi_0^*, \sin \phi_0^*)$ around the origin in the space $X$ when the location $\theta$ changes by $2\pi$. We define the rotation as being positive (negative) when the rotation is anticlockwise (clockwise). In the $P$ solution, all oscillators desynchronize, whereas in the other three solutions, an extensive number of oscillators synchronize and their directions become locked. We depict $\theta$ dependencies of the entrained phase $\phi_0^*$ for each solution in Figs. 1(a)–1(c). Although $\phi_0^*$ fluctuates in all solutions, the behavior of $\phi_0^*$ together with the rotation number characterize each solution.

In the $U$ solution, $\phi_0^*$ randomly takes on a value in the interval $[-\frac{\pi}{2}, \frac{\pi}{2} + \Theta]$ irrespective of the location of oscillators; hence, the rotation number is 0. In the $S$ solution, $\phi_0^*$ linearly depends on $\theta$, and the rotation number is $\pm 1$. See Appendix B for the derivations of these solutions. In the $Pn$ solution, $\phi_0^*$ has an oscillatory behavior and the rotation number is 0, but the directions of neighboring synchronized oscillators are weakly correlated.

**IV. SELF-CONSISTENT EQUATIONS FOR ORDERED SOLUTIONS**

This section uses the same notation as in Sec. II; that is, $\Theta$’s denote $\Theta^*$’s. We give the SCEs and relevant quantities of various solutions of the SCEs. The derivations are given in Appendix B.

**A. Stable U solution**

In the $U$ solution, $R_1 = 0$. This is merely the solution of the Kuramoto model,

\[
R = 2J_0R \int_0^{\pi/2} d\psi \ g(\omega_0 + J_0 R \sin \psi) \cos^2 \psi,
\]

\[
A_\theta = J_0 R, \quad \alpha_0 = \Theta = \text{const},
\]

\[
\phi_\theta^* = \omega_0 \theta + \Theta + \sin^{-1} \left( \frac{\omega_0 - \omega_0}{J_0 R} \right).
\]

The phase transition point from the $P$ phase to the $U$ phase is

\[
J_{0,c} = \frac{2}{\pi g(\omega_0)}.
\]

**B. Stable S solution**

For the stable $S$ solution, $R = 0$, $\langle \cos \theta \rangle = \langle \sin \theta \rangle = 0$, and $\Theta_e - \Theta_s = \pm \frac{\pi}{2}$.

\[
R_c = J_1 R_c \int_0^{\pi/2} d\psi \ g(\omega_0 + J_1 R_c \sin \psi) \cos^2 \psi,
\]

\[
R_c = R_s = \frac{R_1}{\sqrt{2}}, \quad A_\theta = J_1 R_c, \quad \alpha_0 = \Theta = \Theta' \mp \theta,
\]

\[
\phi_\theta^* = \omega_0 \theta + \sin^{-1} \left( \frac{\omega_0 - \omega_0}{J_1 R} \right) + \Theta' \mp \theta.
\]

The phase transition point from the $P$ phase to the $S$ phase and the order parameter $R_1$ near the transition point are

![FIG. 1. $\theta$ dependencies of entrained phase $\phi_0^*$. Line plots: theory: $+$ simulation ($N = 10000$, $\sigma = 0.2$, $J_0 = 1.2J_{0,c}$, $\omega_0 = 0$). Since $\omega_0$ is 0, the theoretical values are calculated by using Eq. (11). These values are connected by straight lines so that it is easier to compare them with the numerical results. Only 1% of the entrained phases are depicted. (a) $U$ solution, $J_1/J_0 = 1.9$; (b) $S$ solution, $J_1/J_0 = 2.1$; (c) $Pn$ solution, $J_1/J_0 = 2.1$.](image)
given by
\[ J_{1,c} = \frac{4}{g(\omega_0)\pi} = 2J_{0,c}, \]
\[ R_1 \approx \frac{4}{J_{c,c}} \sqrt{\frac{2(J_1 - J_{c,c})}{\pi |g'(\omega_0)|}} \propto \sqrt{J_1 - J_{c,c}}. \]  
(30)

When \( g(\omega) \) is a Gaussian distribution with mean \( \omega_0 \) and standard deviation \( \sigma \), \( J_{0,c} \) and \( J_{1,c} \) become
\[ J_{0,c} = 2\sqrt{\frac{2}{\pi}} \sigma, \quad J_{1,c} = 4\sqrt{\frac{2}{\pi}} \sigma = 2J_{0,c}. \]

The entrained phase \( \phi' \) linearly depends on \( \theta \). Since the resultant frequency distributions for the synchronized and desynchronized oscillators do not depend on \( \theta \), we denote them by \( G_s(\tilde{\omega}) \) and \( G_d(\tilde{\omega}) \), respectively,
\[ G_s(\tilde{\omega}) = \frac{N_s}{\sqrt{\tilde{\omega} - \omega_0}}, \]
\[ G_d(\tilde{\omega}) = \frac{1}{\sqrt{\tilde{\omega} - \omega_0}} \frac{1}{\sqrt{\tilde{\omega} - \omega_0}} \sum_{R=1}^{N_s} (\tilde{\omega} - \omega_0)^2 + (J_1 R)^2 \]
\[ \times g(\omega_0 + \sqrt{(\tilde{\omega} - \omega_0)^2 + (J_1 R)^2}), \]
where \( N_s \) is the number of synchronized oscillators.

C. Stable \( Pn \) solution

We define the phase \( \psi \) of \( (R_c, R_p) \) as
\[ R_c = R_1 \cos \psi, \quad R_p = R_1 \sin \psi. \]

Furthermore, defining \( \tilde{\omega}_\theta = \omega_0 - \Theta_c \), we have
\[ A_\theta \cos \tilde{\omega}_\theta = J_1 R_1 \cos(\theta - \psi \cos(\Theta_c - \Theta_\psi)), \]
\[ A_\theta \sin \tilde{\omega}_\theta = -J_0 R \sin(\Theta_c - \Theta_\psi), \]
\[ A_\theta = \sqrt{(J_1 R_1)^2 + (J_1 R_1)^2 \cos^2(\theta - \psi \cos(\Theta_c - \Theta_\psi))}, \]
where \( \cos(\Theta_c - \Theta_\psi) = \pm 1 \) and \( \cos(\Theta_c - \Theta_\psi) = \pm 1 \).

By transforming \( \theta \) into \( \theta' = \theta - \psi \cos(\Theta_c - \Theta_\psi) \), the SCEs become
\[ R = \frac{4J_0 R_1}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} d\theta' g(\omega_0 + A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} \sin \psi) \]
\[ \times \cos^2 \psi, \]
\[ R_1 = \frac{4J_1 R_1}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} d\theta' g(\omega_0 + A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} \sin \psi) \]
\[ \times \cos^2 \psi \cos^2 \theta', \]
\[ A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} = (J_1 R_1)^2 \cos^2 \theta', \]
\[ \phi_\theta* = \omega_0 \theta + \alpha_\theta + \sin^{-1} \left( \omega_0 - \omega_0 \right) \]
\[ = \omega_0 \theta + \tilde{\omega}_\theta + \Theta_c' + \sin^{-1} \left( \omega_0 - \tilde{\omega}_\theta \right). \]
(31)

D. Unstable \( Pn \) solution

Setting \( \alpha_\theta = \omega_0 - \Theta_c \), we have
\[ A_\theta \sin \tilde{\omega}_\theta = -J_1 R_1 \sin(\Theta_c - \Theta_\psi), \]
\[ A_\theta \cos \tilde{\omega}_\theta = J_0 R \cos(\Theta_c - \Theta_\psi) + J_1 R_1 \cos \theta, \]
\[ A_\theta = \sqrt{(J_0 R \cos(\Theta_c - \Theta_\psi) + J_1 R_1 \cos \theta)^2 + (J_1 R_1 \sin \theta)^2}. \]

By transforming \( \theta \) into \( \theta' = \theta - (\Theta_c - \Theta_\psi) \), Eqs. (21), (22), and (23) become
\[ R = \frac{2}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} d\theta' g(\omega_0 + A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} \sin \psi) \cos^2 \psi \]
\[ \times (J_0 R + J_1 R_1 \cos \theta'), \]
\[ R_1 = \frac{2}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} d\theta' g(\omega_0 + A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} \sin \psi) \cos^2 \psi \]
\[ \times (J_0 R + J_1 R_1 \cos \theta'), \]
\[ R_\theta = \frac{2}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} d\theta' g(\omega_0 + A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} \sin \psi) \]
\[ \times \cos^2 \psi \sin^2 \theta', \]
\[ A_{\theta' + \psi \cos(\theta_\psi - \Theta_\psi)} = (J_0 R + J_1 R_1 \cos \theta')^2 + (J_1 R_1 \sin \theta')^2. \]

By numerically solving Eqs. (34) and (35) for the stable \( Pn \) solution, we obtain the \( U \) solution by setting \( R_1 = 0 \) and the stable \( Pn \) solution by setting \( R_1 \neq 0 \). This is because Eq. (34) with \( R_1 = 0 \) reduces to the SCE (27) for the \( U \) solution. A necessary condition to obtain a stable \( Pn \) solution is \( J_1 > J_0 \).

We prove this in Appendix C.

V. PHASE DIAGRAM AND BIFURCATION STRUCTURE

Figure 2(a) is the phase diagram in scaled parameter space \( (J_0/\sigma, J_1/\sigma) \). Here, \( \sigma \) is the standard deviation of the Gaussian distribution \( g(\omega) \) that was used in the simulations. The diagram shows that the \( S \) and \( U \) solutions can coexist and the \( S \) and \( Pn \) solutions can coexist. Figure 2(b) shows the \( J_1/\sigma \) dependencies of the order parameters \( R_1 \) and \( J_1 \) with \( J_0/\sigma \) fixed to 4. From this figure, we can see that the \( Pn \) solution bifurcates from the \( U \) solution continuously, as is proved in Appendix C. On the other hand, the \( S \) solution bifurcates from the \( P \) solution at the critical value of \( J_1/\sigma \). Furthermore, the unstable \( Pn \) solution and stable \( S \) solution merge and the stable \( S \) solution becomes unstable as \( J_1/\sigma \) decreases, and the unstable and stable \( Pn \) solutions merge and the \( Pn \) solution becomes unstable as \( J_1/\sigma \) increases. Therefore, the unstable \( Pn \) solution determines the boundary of the coexisting regions of the \( S \) and \( U \) solutions and of the \( S \) and \( Pn \) solutions.

Taking into account these observations, we can derive the boundaries of the coexisting solutions by using the unstable \( Pn \) solution and relevant stable solutions. The equations for the boundaries are derived in Appendix D.

VI. NUMERICAL RESULTS

We performed numerical simulations using a Gaussian distribution with mean 0 and standard deviation \( \sigma \) as \( g(\omega) \). That is, \( \omega_0 = 0 \). If \( J_0 \) or \( J_1 \) is large in the calculation, the discretization of \( \frac{d\omega}{dt} \) by the Euler method, \( \phi_{i+1} - \phi_i \), becomes worse [6]. Since the evolution equations with the same values
of $\omega_0/\sigma$, $J_0/\sigma$, and $J_1/\sigma$ become identical by changing the time scale from $\tau$ to $\sigma \tau$, we fix $J_0$ or $J_1$ and change $\sigma$ when $J_0$ or $J_1$ is large. The Euler method had a time increment $h = 0.1$.

### A. Phase diagram

Figure 2(a) depicts the theoretically and numerically obtained boundaries. The theoretical results [the solid curves] are in good agreement with the simulation results [symbols]. Now, let us examine the physical meanings of the phase transitions by using the rotation numbers of the solutions. There are five boundaries in the phase diagram shown in Fig. 2(a). The transition from the $P$ to $U$ phase takes place continuously at $J_0/\sigma = (J_0/\sigma)_c$, for $0 \leq J_1/\sigma \leq (J_1/\sigma)_c$, and this is the same transition as in the Kuramoto model. The transition from the $P$ to $S$ phase takes place continuously at $J_1/\sigma = (J_1/\sigma)_c = 2(J_0/\sigma)_c$ for $0 \leq J_0/\sigma \leq (J_0/\sigma)_c$. In the $P$ phase, there are no synchronized oscillators and the rotation number is not defined. In the $S$ phase, the rotation number is 1. The above illustrates that a solution with a nonzero rotation number can appear from a solution in which the rotation number is not defined. Another example of this occurs in model 2 (see Sec. VIII). That is, the $S_m$ solution with the rotation number $\pm m$ appears from the $P$ solution when the $m$th Fourier component exists in the interaction. The transition from the $U$ to $Pn$ phase takes place continuously at $J_1/\sigma = 2J_0/\sigma$ for $J_0/\sigma \geq (J_0/\sigma)_c$, and $J_1/\sigma \geq (J_1/\sigma)_c$. In this case, the rotation number of the $U$ and $Pn$ phases is 0. This is reasonable because the transition is continuous and synchronized oscillators exist in both phases. Although both solutions have the same rotation number, 0, they are different. That is, as is shown in Figs. 1(a) and 1(c), the directions of the two synchronized oscillators do not correlate in the $U$ phase, but they correlate weakly in the $Pn$ phase. Now, let us investigate the transitions at the bistable region boundaries. As mentioned in the previous section, the stable $S$ and unstable $Pn$ solutions merge and the stable $S$ solution disappears at the boundary between the $S$ and $U$ solutions, and the stable $Pn$ and unstable $Pn$ solutions merge and the stable $Pn$ solution disappears at the boundary between the $S$ and $Pn$ solutions. This type of transition does not exist in the Kuramoto model. When the stable $S$ solution with rotation number $\pm 1$ and the unstable $Pn$ solution merge, these two solutions should have the same rotation number, $\pm 1$. Likewise, when the stable $Pn$ solution with rotation number 0 and the unstable $Pn$ solution merge, their rotation numbers should be 0. Thus, the rotation number of the unstable $Pn$ solution changes from $\pm 1$ to 0 as $J_1/\sigma$ increases. This is really the case, as evidenced by Fig. 3. When we calculated $\phi^*_s$ for the unstable solution, we assumed $\Theta_c = 0$, $\Theta_c - \Theta = 0$, and $\Theta_c - \Theta = \pi/2$, and we fixed these values when values of $J_1/\sigma$ change, because these phases take on discrete values and are considered to be continuous with respect to the system parameters. The reason why the rotation number can change as a system parameter changes is as follows. Oscillators are discretely spread out on a circle and the difference between the phases of the neighboring synchronized oscillators can become $\pi$. Since the phase difference is defined in mod $2\pi$, if it changes and takes on the value $\pi$, the rotation number will change by $\pm 1$. The rotation number is not defined if there are neighboring synchronized oscillators whose phases differ by $\pi$. However, this situation is very special. In most cases, it is defined and can be used to classify solutions and get physical information on them.

### B. Spinning solution

Figures 4(a) and 4(b) plot the dependence of $R_1$ and the distribution of the resultant frequencies $G(\omega')$ on $\frac{\omega}{\sigma}$. The location-dependent entrained phase $\phi^*_c$ is in Fig. 1(b). It turns out that the entrained phase depends linearly on $\theta$ in the $S$ solution but takes on random values for the $U$ solution [see Fig. 1(b)], as theoretically expected. All of the numerical results agree quite well with the theoretical results.

### C. Pendulum solution

Figure 5 displays the theoretical and simulated results of the $J_0$ dependence of the order parameters for $J_1 = 2.1J_0$. The
FIG. 3. Theoretical estimation of $\theta$ dependencies of phases $\phi^*_\theta$ for unstable $Pn$ solution. $\sigma = 0.2$, $J_0/\sigma = 4$. The theoretical values are calculated using a similar equation to Eq. (11) for the unstable $Pn$ solution, and these values are connected by straight lines. Only 1% of the phases are depicted. (a) $J_1/\sigma = 6$, near the boundary of the region in which the $S$ and $U$ phases coexist; (b) $J_1/\sigma = 8$, at the boundary of the region in which the $S$ and $U$ phases coexist; (c) $J_1/\sigma = 10$, near the boundary of the region in which the $S$ and $Pn$ phases coexist.

Theoretical and simulation results of the location-dependent resultant frequency distribution $G(\tilde{\omega}, \theta)$ for different $\theta$ are in Fig. 6, and those of the $\theta$ dependencies of the entrained phases $\phi^*_\theta$ are in Fig. 1(c). The agreement between the theoretical and numerical results is excellent. To investigate the desynchronized oscillators, we constructed a Lorenz plot of the time series $\sin[\varphi_i(t)]$ for the $Pn$ solution (Fig. 7).

The Lorenz plot is a mapping from the difference $\Delta t_i = t_{i+1} - t_i$ to $\Delta t_{i+1}$, where $t_i$ and $t_{i+1}$ are successive times that satisfy $\cos[\varphi_i(t_i)] = 1$ and $\cos[\varphi_i(t_{i+1})] = 1$, respectively. As shown in Fig. 7, the simulation results are scattered in the Lorenz plot. This indicates that the trajectory of a desynchronized oscillator behaves chaotically even though theoretically it is quasiperiodic. However, this is reasonable because synchronized and desynchronized oscillators interact with other oscillators, and desynchronized oscillators are easily influenced by perturbations, whereas entrained oscillators are forced to lock to the fixed phases. As is well known, quasiperiodic motion with more than two dimensions generically becomes chaotic through perturbation [28]. In most of our numerical results, e.g., those for the resultant frequency distribution, the larger $N$ is, the better the agreement between the theoretical and numerical results becomes. These results suggest that the system behaves quasiperiodically as $N$ goes to infinity.

VII. SYSTEM WITH INTERACTION COMPOSED OF FIRST AND SECOND FOURIER COMPONENTS

In this section, we study model 2, in which the interaction is given by

$$J_{\theta, \theta'} = \frac{1}{N} [J_m \cos(m(\theta - \theta')) + J_n \cos(n(\theta - \theta'))].$$  (41)
The order parameters are defined as $\Delta_0 \equiv \frac{1}{N} \sum_{\theta} \cos(k\theta)e^{i\phi_0}$. For later use, we define $\psi_\theta = \phi_\theta - \omega_\theta t - \alpha_\theta$, the evolution equation becomes
\[
\frac{d}{dt} \psi_\theta = \omega_\theta - \omega_0 - A_\theta \sin \psi_\theta.
\]

Below, for simplicity, we will omit primes from the phases except for the expressions of $\alpha_\theta$ and $\phi_\theta^*$. We give the SCEs for the stable spinning pendulum solutions. The derivations are in Appendix E. Below, for simplicity, we will omit primes from the phases except for the expressions of $\alpha_\theta$ and $\phi_\theta^*$.

### A. Spinning solution

We give the conditions and SCEs for the stable spinning solution with $R_m > 0$ and $R_n = 0$. We denote the solution by $S_m$. The conditions on this solution are $\cos(\Theta_{mc} - \Theta_{ms}) = 0$ and $\cos(2m\phi_{mc}) = 0$. From this, $R_{mc} = R_{ms} = \frac{1}{2}R_3$ and $A_\theta = J_m R_{mc}$ follow. The SCE is
\[
1 = J_m \int_0^{\pi/2} d\psi \ g(\omega_0 + J_m R_{mc} \sin \psi) \cos^2 \psi.
\]
This equation is the same as Eq. (29) for the stable spinning solution studied in model 1. The critical point is given by

$$J_m^{(c)} = J_1^{(c)} = \frac{4}{\pi g(\alpha_0)} = 2J_0^{(c)}. \quad (47)$$

That is, the critical point for \( S_m \) is the same as the one for the \( S \) solution in model 1. The synchronized solution is expressed as

$$\alpha_0 = \theta'_0 \mp mt \phi,$$

$$\phi_0^* = \omega_0 t + \psi_0^* + \alpha_0$$

$$= \omega_0 t + \sin^{-1}\left(\frac{\alpha_0 - \alpha_0}{J_m R_m} + \theta'_0 \mp m \theta.\right) \quad (48)$$

The entrained phase of the solution changes by \( \pm 2\pi m \) when the location \( \theta \) changes by \( 2\pi \); that is, its rotation number is \( \pm m \).

### B. Pendulum solution

Here, we assume \( R_m R_n \neq 0 \). Moreover, for simplicity, we assume \( n \neq 3m \). Accordingly, the conditions on the stable \( Pn \) solution are

$$\sin(\Theta_{mc} - \Theta_{mr}) = 0, \quad \sin(\Theta_{nc} - \Theta_{mr}) = 0,$$

$$\cos(\Theta_{nc} - \Theta_{mr}) = 0. \quad (49)$$

Here, we define

$$\tilde{\theta}_k = \phi_k e^{i(\theta_{mc} - \theta_{mc})},$$

$$\tilde{\theta} = \tilde{\theta}_n - \tilde{\theta}_m, \quad \theta' = \tilde{\theta} - \tilde{\theta}_m,$$

from which we obtain

$$A_\theta = A_{\theta' + \tilde{\theta}_m} = \sqrt{[J_m R_m \cos(m\theta')]^2 + [J_n R_n \cos(n\theta' - \tilde{\theta})]^2}. \quad (50)$$

By changing the variable from \( \theta \) to \( \theta' \), the SCEs become

$$1 = \frac{J_m}{\pi} \int_{\theta_1}^{\theta_2} d\theta \int_0^{2\pi} d\psi g(\omega_0 + A_{\theta' + \tilde{\theta}} \sin \psi) \times \cos^2 \psi \cos^2 (m\theta'), \quad (51)$$

$$1 = \frac{J_n}{\pi} \int_{\theta_1}^{\theta_2} d\theta \int_0^{2\pi} d\psi g(\omega_0 + A_{\theta' + \tilde{\theta}} \sin \psi) \times \cos^2 \psi \cos^2 [n(\theta' - \tilde{\theta})]. \quad (52)$$

Furthermore, we derive the condition

$$\langle \sin[2n(\theta' - \tilde{\theta})] \rangle = 0. \quad (53)$$

The sufficient condition for this is \( \cos(2n\tilde{\theta}) = 0 \) or \( \sin(2n\tilde{\theta}) = 0 \), and it determines the value of \( \tilde{\theta} \). Setting \( \tilde{\theta} = \alpha_0 - \Theta_{mr} \), we obtain

$$A_\theta \cos \tilde{\theta} = J_m R_m \cos(\Theta_{mc} - \Theta_{mr}) \cos(m\theta'), \quad (54)$$

$$A_\theta \sin \tilde{\theta} = J_n R_n \sin(\Theta_{nc} - \Theta_{mr}) \cos(n(\theta' - \tilde{\theta})). \quad (55)$$

Thus,

$$\phi_0^* = \omega_0 t + \sin^{-1}\left(\frac{\omega_0 - \omega_0}{A_\theta} + \tilde{\theta} + \phi_{kc} \right) \quad (56)$$

This formula indicates that the rotation number can take on values of \( \pm 1, \pm 2, \ldots, \pm m \).

### C. Numerical results

We performed numerical simulations in which we used a Gaussian distribution for \( g(\omega) \). We set the mean to be 0, i.e., \( \omega_0 = 0 \) and standard deviation \( \sigma \). We studied the case of \( m = 1 \) and \( n = 2 \). Figure 8 plots the time series of the amplitudes of the complex order parameters for \( S_1 \), \( S_2 \), and \( Pn \). The trajectories of the amplitudes of the complex order parameters for \( S_1 \) and \( S_2 \) converge, but those for \( Pn \) fluctuate. Figure 9 shows the time series of the phases of the complex order parameters and \( 4\tilde{\theta} \). Moreover, Figs. 10 and 11, respectively, display the \( J_2 \) dependence of the order parameters for \( J_1 = J_2 \) and the \( \theta \) dependencies of the entrained phases \( \phi_0^* \). The rotation numbers for the \( S_1 \), \( S_2 \), and \( Pn \) solutions are \( -1, 2, \) and \( 0 \), respectively. The theoretical and numerical results agree quite well. Figure 12 shows the resultant frequency distribution \( G(\tilde{\omega}) \) of the \( S_1 \) and \( S_2 \) solutions, and the location-dependent resultant frequency distribution \( G(\tilde{\omega}, \theta) \) for different \( \theta \) of the \( Pn \) solution. Agreement between numerical and theoretical results is excellent for the \( S_1 \) and \( S_2 \) solutions. However, it is not close for the \( Pn \) solution. This is because the trajectories of \( R_1 \) and \( R_2 \) for the \( Pn \) solution fluctuate, as shown in Fig. 8(c). As can be seen from Fig. 9, \( \Theta_{1c} - \Theta_{1r} = -\pi, \Theta_{2c} - \Theta_{2r} = 0, \Theta_{2c} - \Theta_{1r} = -\pi/2 \). Thus, \( \tilde{\theta}_1 = -\phi_1 \) and \( \tilde{\theta}_2 = \phi_2 \). Therefore, \( 0 \leq \tilde{\theta} \leq 3\pi/2 \). Accordingly,
the condition $\cos(4\tilde{\theta}) = 0$ implies $4\tilde{\theta} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$. The other condition $\sin(4\tilde{\theta}) = 0$ implies $4\tilde{\theta} = 0, \pi, 2\pi, 3\pi$. However, numerical results show that $4\tilde{\theta} \simeq 2.2\pi$ [Fig. 9(c)]. This is the cause of the discrepancy between the theoretical and numerical results for $G(\tilde{\omega}, \theta)$.

VIII. SUMMARY AND DISCUSSION

We studied phase oscillator networks on a circle with two types of interaction (models 1 and 2).

Model 1 is the Mexican-hat interaction. The interaction is composed of two terms, one of which is a uniform interaction with strength $J_0$, and the other is a sinusoidal interaction with respect to the location $\theta$ of oscillators with strength $J_1$. If $J_1 = 0$, the present model reduces to the Kuramoto model. To obtain self-consistent equations, information about the differences between the phases of the complex order parameters is necessary. Previously, for this purpose, we studied the classical $XY$ model to which the phase oscillator network reduces when all oscillators have the same natural frequency [29,30]. The relevant order parameters are the same in the oscillator network and the $XY$ model. The order parameters that characterize the solutions are $R$ and $R_1$. The saddle point equations (SPEs) for the $XY$ model were obtained and the differences between the phases of the complex order parameters were determined analytically [30]. So far, we have used information on the phases of the complex order parameters in the $XY$ model to solve the SCEs for the phase oscillator network. Recently, we succeeded in deriving auxiliary equations that determine the phases of the order parameters for the phase oscillator network. The auxiliary equations turned out to be the same as the SPEs for the phases of the complex order parameters in the $XY$ model [26]. In this paper, we gave detailed derivations of the relevant equations and quantities. We derived two auxiliary equations by expressing the order parameters in terms of the number density of the oscillators. We used them to analytically determine the phases of the order parameters, derived self-consistent equations for their amplitudes, and obtained three nontrivial solutions that are characterized by the order parameters and the rotation numbers of the synchronized oscillators $X_{\theta}^*$. We drew the phase diagram by using formulas for the phase boundaries derived using the unstable $Pn$ solution. Furthermore, we found that the disappearance of coexistent regions between the $U$ and $S$ solutions and between the $Pn$ and $S$ solutions is due to annihilation of the unstable $Pn$ and stable $S$ solutions, and that of the unstable and stable $Pn$ solutions. This type of transition does not exist in the Kuramoto model. We also analytically obtained the location-dependent distribution of the resultant frequencies and entrained phases and validated the theoretical results by simulation, except for the chaotic behavior of the desynchronized oscillators. This chaotic behavior is quite reasonable because quasiperiodic motion with more than two dimensions generically becomes chaotic through perturbation.
FIG. 11. θ dependencies of entrained phases $\phi_0^*$. Line plots: theory; +: simulation. The theoretical values are calculated using Eq. (56) with $\omega_0 = 0$, and these values are connected by straight lines so that it is easier to compare them with the numerical results. Only 1% of the entrained phases are depicted. $J_1 = J_2 = 1.8 J_{1,c}, \sigma = 0.2$. Simulation: $N = 10000$. (a) $S_1$ solution. (b) $S_2$ solution. (c) $P_n$ solution.

[28]. Our numerical results suggest that the system behaves quasiperiodically as $N$ goes to infinity.

![Graphs showing theoretical and numerical results for $\phi_0^*$ and $G(\omega)$](image)

FIG. 12. Theoretical and simulated results for the distribution of the resultant frequencies. $G(\omega)$ for the $S_1$ and $S_2$ solutions and $G(\omega, \theta)$ for the $P_n$ solution. Curve: theory; +: simulation ($N = 200000, \sigma = 0.01, J_1 = J_2 = 1.2 J_{1,c}$). (a) $S_1$ solution, (b) $S_2$ solution, (c) $\theta = 0.05 \times 2\pi$ for $P_n$ solution, (d) $\theta = 0.25 \times 2\pi$ for $P_n$ solution.
and Sm and Sn solutions are the same as those for the S solution in model 1. In Sn (Sm), the phase of the entrained oscillators changes by ±2nπ (±2nπ) when the location changes by 2π. That is, the rotation number is ±m (±n). On the other hand, in the pendulum solution, the phase of the entrained oscillators fluctuates when location changes by 2π, and the rotation number can take on values of 0, ±1, ±2, . . . , ±m.

We performed a similar analysis to that of model 1 for m = 1 and n = 2. In the J1, J2 space, around the line J1 = J2, these three solutions coexist for J1 > Jc and J2 > Jc. We conducted a simulation that validated the theoretical results for the J2 dependencies of R1 and R2 and the dependencies of the entrained phases on location. As for the distribution of the resultant frequencies, the agreement between the theoretical and numerical results was excellent for the S1 and S2 solutions. However, theoretical and numerical results for the location-dependent distribution of the resultant frequencies did not agree very well for the Pn solution. This is because the trajectories of R1 and R2 for the Pn solution fluctuate, and the numerical value of ˜θ deviates from the theoretical prediction. The reason for this deviation seems to be that the desynchronized oscillators behave more chaotically in the Pn solution than in the S1 and S2 solutions.

How the existing phases and phase transitions depend on the type of interaction is an interesting question. The present method is applicable to phase oscillator networks not only on a circle but also in general spaces. For example, we have started to study an oscillator network and XY model in which the interaction is like that of associative memory, and we have found that there are different phases and phase transitions from those in the Mexican-hat interaction.

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APPENDIX A: DERIVATION OF AUXILIARY EQUATIONS AND SCES, AND SOLUTIONS OF AUXILIARY EQUATIONS

The total number density of oscillators with phase ψ at location θ, n(θ, ψ), is n(θ, ψ) = ns(θ, ψ) + nd(θ, ψ). From Eq. (15), since nd(θ, ψ + π) = nd(θ, ψ), ∫π −π nd(θ, ψ)eiθ dθ = 0 follows. Thus, only synchronized oscillators contribute to the order parameters:

\[ R_{e^{iθ}} = \int_0^{2π} dψ n_s(ψ) e^{iψ + iα_v}, \]  

\[ R_{e^{iθ}} = \int_0^{2π} dψ \frac{1}{2π} \int_0^{2π} dθ n_d(θ, ψ) cos \theta e^{iψ + iα_v}, \]  

\[ R_{e^{iθ}} = \int_0^{2π} dψ \frac{1}{2π} \int_0^{2π} dθ n_d(θ, ψ) sin \theta e^{iψ + iα_v}, \]  

where

\[ n_s(ψ) = \frac{1}{2π} \int_0^{2π} dθ n_s(θ, ψ). \]

By substituting the expression of ns(θ, ψ) into Eq. (A1), we obtain the following equations:

\[ R = \frac{1}{π} \int_0^{2π} dψ \int_0^{2π} dθ g(ω_0 + A_θ sin ψ) cos^2 ψ \times \{ J_0 R + J_1 (R_c cos θ e^{i(θ, −θ_0)} + R_s sin θ e^{i(θ, −θ_0)}). \]  

(A4)

Aθ is expressed as

\[ A_θ^2 = (J_0 R)^2 + J_1^2 (R_c cos θ)^2 + (R_s sin θ)^2 + 2 J_0 J_1 R_c (R_c cos(θ_1 − θ_2) sin θ) \]  

+ 2 J_0 J_1 R_c (R_c cos(θ_1 − θ_2) sin θ) \]  

+ 2 J_0 J_1 R_c (R_c cos(θ_1 − θ_2) sin θ) \]  

+ 2 J_0 J_1 R_c (R_c cos(θ_1 − θ_2) sin θ).

We introduce the following notation:

\[ \langle B \rangle = \frac{1}{Z} \int_0^{2π} dψ \int_0^{2π} dθ g(ω_0 + A_θ sin ψ) cos^2 ψ B, \]  

(A5)

\[ Z = \int_0^{2π} dψ \int_0^{2π} dθ g(ω_0 + A_θ sin ψ) cos^2 ψ. \]  

(A6)

Equation (A4) can be rewritten as

\[ R = (J_0 R + J_1 (R_c e^{i(θ, −θ_0)}(cos θ) + R_s e^{i(θ, −θ_0)}⟨sin θ⟩)) Z. \]  

(A7)

Similarly, Eqs. (A2), and (A3) can be rewritten as

\[ R_c = (J_0 R e^{-i(θ, −θ_0)}(cos θ) + J_1 (R_0 cos^2 θ) + R_s ⟨sin θ cos θ⟩) Z. \]  

(A8)

\[ R_s = (J_0 R e^{-i(θ, −θ_0)}(sin θ) + J_1 (R_s e^{i(θ, −θ_0)}⟨sin θ cos θ⟩) + R_s (sin^2 θ)) Z. \]  

(A9)

The real parts of Eqs. (A7), (A8), and (A9) give the SCEs for R, Rc, and Rs,

\[ R = (J_0 R + J_1 (R_c cos θ)(cos(θ_2 − θ))) Z, \]  

(A10)

\[ R_c = (J_0 R cos θ)(cos(θ_2 − θ) + J_1 (R_0 cos^2 θ) + R_s (sin θ cos θ)(cos(θ_2 − θ))) Z, \]  

(A11)

\[ R_s = (J_0 R (sin θ)(cos(θ_2 − θ) + J_1 (R_s (sin θ cos θ) × cos(θ_2 − θ)) + R_s (sin^2 θ)) Z. \]  

(A12)

The imaginary parts of Eqs. (A7), (A8), and (A9) give three equations,

\[ R_c (cos θ)(sin(θ_2 − θ) + R_s (sin θ)(sin(θ_2 − θ)) = 0, \]  

(A13)

\[ J_0 R (cos θ)(sin(θ_2 − θ)) \]  

+ J_1 R_s (sin θ cos θ) sin(θ_2 − θ) = 0, \]  

(A14)

\[ J_0 R (sin θ)(sin(θ_2 − θ)) \]  

− J_1 R_s (sin θ cos θ) sin(θ_2 − θ) = 0, \]  

(A15)

Two of Eqs. (A13), (A14), and (A15) are independent. These auxiliary equations completely determine the phases of the order parameters.
Now, let us solve the auxiliary equations. We will concentrate on the solutions that are relevant to the phase transitions. First, we solve the case of $R = 0$ and $R_1 = \sqrt{R_c^2 + R_s^2} \neq 0$ and derive the phases of the order parameters for the stable $S$ solution.

Case of $R = 0, R_1 \neq 0$. Since $A_0$ does not have $\cos \theta$ and $\sin \theta$ terms, $\langle \cos \theta \rangle = \langle \sin \theta \rangle = 0$ follows. Thus, from Eqs. (A14) and (A15), we obtain

$$J_1 R_c \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s) = 0, \quad (A16)$$

$$J_1 R_c \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s) = 0. \quad (A17)$$

Therefore, we have

$$\langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s) = 0.$$  

The case of $\langle \sin \theta \cos \theta \rangle = 0$ gives an irrelevant solution, so we will omit discussion of this case.

Now let us study the case $\langle \sin \theta \cos \theta \rangle = 0$. Here, the Fourier expansion of the integrand should not contain the Fourier component $\sin \theta \cos \theta$. Therefore, the coefficient of $\sin \theta \cos \theta$ in $A_0$ should be 0, that is, $R_c R_s \cos(\Theta_c - \Theta_s) = 0$. Thus, $\cos(\Theta_c - \Theta_s) = 0$, or $R_c = 0$, or $R_s = 0$. $R_1$ is irrelevant, and the relevant solution is when $\cos(\Theta_c - \Theta_s) = 0$, that is, $\Theta_c - \Theta_s = \pm \frac{\pi}{2} \text{ (mod } 2\pi\text{)}$. Hereafter, we omit "(mod $2\pi$)" for simplicity. Numerical results show that this case corresponds to the stable $S$ solution.

Next, we solve the case of $R \neq 0, R_1 \neq 0, R_s \neq 0$ and derive the phases of the order parameters for the $Pn$ solutions.

Case of $R \neq 0, R_1 \neq 0, R_s \neq 0$. Using Eq. (A14) and $\Theta_c - \Theta = (\Theta_c - \Theta_s) - (\Theta_s - \Theta)$ and multiplying Eq. (A15) by $\langle \cos \theta \rangle$, we obtain

$$\sin(\Theta_c - \Theta_s)[J_1 R_c \langle \sin \theta \cos \theta \rangle \langle \sin \theta \rangle \cos(\Theta_c - \Theta_s)$$

$$\langle \cos \theta \rangle [J_1 R_c \langle \sin \theta \cos \theta \rangle \cos(\Theta_c - \Theta)$$

$$+ J_1 R_c \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s)] = 0. \quad (A18)$$

Case 1 $\sin(\Theta_c - \Theta_s) = 0$, that is, $\Theta_c - \Theta_s = \{0, \pi\}$. From Eq. (A14), we obtain

$$J_0 R_c \langle \cos \theta \rangle \sin(\Theta_c - \Theta) = 0.$$  

Thus, $\Theta_c - \Theta = \{0, \pi\}$ or $\langle \cos \theta \rangle = 0$ follows. The relevant solution is obtained from the case $\langle \cos \theta \rangle = 0$. In this case, $A_0$ should not contain any $\cos \theta$ term, that is, $R_c R_s \cos(\Theta_c - \Theta_s) \langle \sin \theta \cos \theta \rangle = 0$. Therefore, $\cos(\Theta_c - \Theta) = 0$, i.e., $\Theta_c - \Theta = \pm \frac{\pi}{2}$. Since $\Theta_c - \Theta = (\Theta_c - \Theta_s) - (\Theta_s - \Theta)$, we obtain

$$\{\Theta_c - \Theta, \Theta_s - \Theta\} = \left\{\pm \frac{\pi}{2}, \pm \frac{\pi}{2}\right\}.$$  

Numerical results show that this is the stable $Pn$ solution. Using Eq. (A13) and $\Theta_c - \Theta = (\Theta_c - \Theta_s) - (\Theta_s - \Theta)$ and multiplying Eq. (A14) by $\langle \sin \theta \rangle$, we obtain

$$\sin(\Theta_c - \Theta_s)J_0 R_c \langle \cos \theta \rangle$$

$$+ J_1 R_c \langle \sin \theta \cos \theta \rangle \cos(\Theta_c - \Theta)$$

$$= 0. \quad (A19)$$

Case 2 $\sin(\Theta_c - \Theta_s) = 0$, that is, $\Theta_c - \Theta = \{0, \pi\}$. From Eq. (A13), $R_s \langle \sin \theta \rangle \sin(\Theta_c - \Theta) = 0$. Thus, $\Theta_c - \Theta = \{0, \pi\}$, or $\langle \sin \theta \rangle = 0$. The relevant solution is obtained from $\langle \sin \theta \rangle = 0$. In this case, $R_c R_s \cos(\Theta_s - \Theta)$ should be 0. Therefore, $\cos(\Theta_s - \Theta) = 0$, that is, $\Theta_s - \Theta = \pm \frac{\pi}{2}$. Thus, we obtain

$$\{\Theta_c - \Theta, \Theta_s - \Theta\} = \left\{0 \text{ or } \pi, \pm \frac{\pi}{2}\right\}.$$  

It turns out that this $Pn$ solution is unstable.

Using Eq. (A13) and $\Theta_c - \Theta = (\Theta_c - \Theta_s) - (\Theta_s - \Theta)$ and multiplying Eq. (A15) by $\langle \cos \theta \rangle$, we obtain

$$\sin(\Theta_c - \Theta)\langle \cos \theta \rangle [J_0 R_c \langle \sin \theta \cos \theta \rangle$$

$$+ J_1 R_s \langle \sin \theta \cos \theta \rangle \cos(\Theta_c - \Theta_s)$$

$$+ J_1 R_s \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s)] = 0. \quad (A20)$$

Case 3 $\sin(\Theta_c - \Theta_s) = 0$, that is, $\Theta_c - \Theta = \{0, \pi\}$. In this case, from Eq. (A13), we have

$$R_c \langle \cos \theta \rangle \sin(\Theta_c - \Theta) = 0.$$  

It follows that $\Theta_c - \Theta = \{0, \pi\}$, or $\langle \cos \theta \rangle = 0$.

Case 3-1. $\sin(\Theta_c - \Theta) = 0$.

Moreover, it follows that $\{\Theta_c - \Theta, \Theta_s - \Theta\} = \{0 \text{ or } \pi\}$, or $\langle \cos \theta \rangle = 0$. These solutions are unstable and irrelevant.

Case 3-2. $\langle \cos \theta \rangle = 0$.

In this case, $R_c \langle \cos \theta \rangle \cos(\Theta_c - \Theta_s)$ should be 0. Thus, $\cos(\Theta_c - \Theta) = 0$, and $\Theta_c - \Theta = \pm \frac{\pi}{2}$. Therefore, we have $\{\Theta_c - \Theta, \Theta_s - \Theta\} = \{\pm \frac{\pi}{2}, 0 \text{ or } \pi\}$. These solutions are unstable.

Other cases.

The other conditions for (A18), (A19), and (A20) are

$$J_1 R_s \langle \sin \theta \cos \theta \rangle \langle \sin \theta \rangle \cos(\Theta_c - \Theta_s)$$

$$+ \langle \cos \theta \rangle [J_0 R_c \langle \sin \theta \rangle \cos(\Theta_c - \Theta)$$

$$+ J_1 R_s \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s)] = 0, \quad (A21)$$

$$\langle \sin \theta \rangle [J_0 R_c \langle \cos \theta \rangle + J_1 R_s \langle \sin \theta \cos \theta \rangle \cos(\Theta_s - \Theta)$$

$$+ J_1 R_s \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s)] = 0, \quad (A22)$$

$$\langle \cos \theta \rangle [J_0 R_c \langle \sin \theta \rangle + J_1 R_s \langle \sin \theta \cos \theta \rangle \cos(\Theta_s - \Theta)$$

$$+ J_1 R_s \langle \sin \theta \cos \theta \rangle \sin(\Theta_c - \Theta_s)] = 0. \quad (A23)$$

Since $R_c$, $R_s$, and $R_c$ are continuous functions with respect to the parameters $J_0$ and $J_1$, the conditions under which the above equalities hold are that the coefficients of $R_c R_s$, $R_c$ are 0. The conditions for $\langle \sin \theta \rangle = 0$ and for $\cos(\Theta_c - \Theta_s) = 0$ are the same, and $\Theta_c - \Theta = \pm \frac{\pi}{2}$. Similarly, the condition for $\langle \sin \theta \rangle = 0$ and for $\cos(\Theta_s - \Theta) = 0$ is $\Theta_s - \Theta = \pm \frac{\pi}{2}$, and the condition for $\langle \cos \theta \rangle = 0$ and for $\cos(\Theta_c - \Theta_s) = 0$ is $\Theta_c - \Theta_s = \pm \frac{\pi}{2}$. The combination of phases of the order parameters is the same as in cases 1 to 3.

APPENDIX B: DERIVATION OF CONCRETE SCEs AND RELEVANT QUANTITIES FOR EACH PHASE

1. Stable uniform solution

This is the case of $R > 0$ and $R_1 = 0$. This is merely the solution of the Kuramoto model. Since $R_1 = 0$, we have

$$A_0 e^{i \omega} = J_0 R e^{i \phi}, \quad A_0 = J_0 R, \quad \omega = \phi' = \text{const.}$$
Therefore, the entrained phase $\phi_0^*$ is expressed as
\[
\phi_0^* = \omega_0 t + \Theta' + \sin^{-1}\left(\frac{\omega_0 - \omega_0}{J_0 R}\right).
\]
Note that $\Theta' = \Theta - \omega_0 t$ is used in the expressions of $\alpha_0$ and $\phi_0^*$. The SCE is
\[
R = 2J_0 R \int_0^{\pi/2} d\psi \ g(\omega_0 + J_0 R \sin \psi) \cos^2 \psi. \quad (B1)
\]
The phase transition point from the $P$ phase to the $U$ phase is
\[
J_{0,c} = \frac{2}{\pi g(\omega_0)}. \quad (B2)
\]

2. Stable $S$ solution

For the stable $S$ solution, $R = 0$, $\langle \cos \theta \rangle = \langle \sin \theta \rangle = 0$, and $\Theta_c - \Theta = \pm \frac{\pi}{2}$ (see Appendix A). In this case, $A_0$ is expressed as $A_0 = J_1 \sqrt{R_c \cos^2 \theta + (R_c \sin \theta)^2}$. From Eqs. (A11) and (A12), we have
\[
R_c = J_1 R_c (\cos^2 \theta) Z, \quad R_s = J_1 R_s (\sin^2 \theta) Z.
\]
If $R_c R_s \neq 0$, then $\langle \cos^2 \theta \rangle = \langle \sin^2 \theta \rangle$ holds. $R_c = R_s = R_0$, follows from this. Accordingly, $A_0 = J_1 R_c$, and the SCE becomes
\[
R_c = J_1 R_c \int_0^{\pi/2} d\psi \ g(\omega_0 + J_1 R_c \sin \psi) \cos^2 \psi. \quad (B3)
\]
The phase transition point from the $P$ phase to the $S$ phase and the order parameter $R_c$ near to the transition point are given by
\[
J_{1,c} = \frac{4}{g(\omega_0) \pi} = 2J_{0,c}, \quad R_1 \simeq \frac{4}{J_{1,c}} \sqrt{\frac{2(J_1 - J_{1,c})}{\pi |g'(\omega_0)|}} \propto \sqrt{J_1 - J_{1,c}}.
\]
When $g(\omega)$ is a Gaussian distribution with mean $\omega_0$ and standard deviation $\sigma$, $J_{0,c}$ and $J_{1,c}$ are given by
\[
J_{0,c} = 2 \sqrt{\frac{2}{\pi} \sigma}, \quad J_{1,c} = 4 \sqrt{\frac{2}{\pi} \sigma} = 2J_{0,c}. \quad (B4)
\]
Let us study the entrained phase $\phi_0^* = \omega_0 t + \psi_0^* + \alpha_0$. Since $\Theta_c - \Theta = \pm \frac{\pi}{2}$, $\Theta_c = \Theta_c - (\Theta_c - \Theta_c) = \Theta_c \mp \frac{\pi}{2}$. From Eq. (8), we have
\[
A_0 e^{i\omega_0 r} = J_1 R_c e^{i\psi_0^*} (\cos \Theta + \sin \Theta e^{\pm i\frac{\pi}{2}}) = J_1 R_c e^{i(\Theta + \Theta')}. \quad (B5)
\]
Therefore, $A_0 = J_1 R_c$ and $\alpha_0 = \Theta' \mp \Theta$. Accordingly, the entrained phase $\phi_0^*$ is
\[
\phi_0^* = \omega_0 t + \psi_0^* + \alpha_0
\]
\[
= \omega_0 t + \Theta' + \sin^{-1}\left(\frac{\omega_0 - \omega_0}{J_1 R_c}\right) + \Theta' \mp \Theta. \quad (B6)
\]
Thus, the entrained phase $\phi_0^*$ linearly depends on the location $\theta$.

3. Stable $P_1$ solution

Since $\{\Theta_c - \Theta, \Theta_c - \Theta\} = \{0, \pm \frac{\pi}{2}\}$, by using $\sin(\Theta_c - \Theta) = \pm 1$, $\cos(\Theta_c - \Theta) = \pm 1$, and $\sin(\Theta_c - \Theta) = 0$, Eq. (8) reduces to
\[
A_0 e^{i\omega_0} = J_1 R_c e^{i(\Theta, -\Theta)} + J_1 R_c \cos \theta,
\]
\[
+ J_1 R_c e^{i(\Theta, -\Theta)} \cos(\theta - \Phi) + i J_1 R_c \sin(\theta - \Phi), \quad (B7)
\]
where we have defined
\[
R_c = R_1 \cos \phi, \quad R_s = R_1 \sin \phi.
\]
Therefore, setting $\alpha_0 = \Theta - \Theta_c$, we have
\[
A_0 \sin \alpha_0 = -J_0 R \sin(\Theta_c - \Theta),
\]
\[
A_0 \cos \alpha_0 = J_1 R_1 \cos(\theta - \Phi \cos(\Theta_c - \Theta)), \quad (B8)
\]
Equations (A10), (A11), and (A12) become
\[
R = J_0 R Z, \quad (B9)
\]
\[
R_c = J_1 R_1 Z \cos(\Phi \cos(\Theta_c - \Theta)) \sin \phi, \quad (B10)
\]
\[
R_s = J_1 R_1 Z \cos(\Phi \cos(\Theta_c - \Theta)) \sin \phi. \quad (B11)
\]
By transforming $\theta$ into $\theta' = \theta - \Phi \cos(\Theta_c - \Theta)$ and since $A_0 \cos(\Phi \cos(\Theta_c - \Theta))$ is periodic with period $\pi$ and an even function of $\theta'$, we obtain the following SCES:
\[
R = \frac{4J_0 R}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} \sin(\theta) \cos^2 \psi, \quad (B12)
\]
\[
R_1 = \frac{4J_1 R_1}{\pi} \int_0^{\pi/2} d\psi \int_0^{\pi/2} \sin(\theta) \cos^2 \psi, \quad (B13)
\]
\[
A_0 \cos(\Theta_c - \Theta) = \sqrt{(J_0 R_c)^2 + (J_1 R_1)^2 \cos^2 \theta'). \quad (B14)
\]
In this solution, we have
\[
\phi_0^* = \omega_0 t + \alpha_0 + \sin^{-1}\left(\frac{\omega_0 - \omega_0}{A_0}\right) = \omega_0 t + \alpha_0 + \Theta_c
\]
\[
\mp \Theta \quad (B15)
\]
where $\theta$ is the original coordinate and $A_0$ is given by Eq. (B8).

4. Unstable $P_1$ solution

Since $\{\Theta_c - \Theta, \Theta_c - \Theta\} = \{0, \pm \frac{\pi}{2}\}$, then $\sin(\Theta_c - \Theta) = \pm 1$, $\sin(\Theta_c - \Theta) = \pm 1$, $\cos(\Theta_c - \Theta) = 0$, $\sin(\Theta_c - \Theta) = 0$, and $\cos(\Theta_c - \Theta) = 0$. Equation (8) reduces to
\[
A_0 e^{i\omega_0} = e^{i(\Theta, -\Theta)} [J_0 R_c e^{i(\Theta, -\Theta)} + J_1 R_c \cos \theta]
\]
\[
- i J_1 R_c \sin \theta \sin(\Theta_c - \Theta), \quad (B7)
\]
Therefore, setting $\theta_0 = \alpha_i - \Theta_c$, we have

\begin{align*}
A_0 \sin \theta_0 &= -J_1 R_c \sin \theta (\Theta_c - \Theta), \\
A_0 \cos \theta_0 &= J_0 R \cos (\Theta_c - \Theta) + J_1 R_c \cos \theta, \\
A_0 &= \sqrt{[J_0 R \cos (\Theta_c - \Theta) + J_1 R_c \cos \theta]^2 + (J_1 R_c \sin \theta)^2}.
\end{align*}

Accordingly, Eqs. (A10), (A11), and (A12) become

\begin{align*}
R &= \langle J_0 R + J_1 R_c \cos (\theta - (\Theta_c - \Theta)) \rangle Z, \quad (B17) \\
R_c &= \langle [J_0 R + J_1 R_c \cos (\theta - (\Theta_c - \Theta))] \times \cos (\theta - (\Theta_c - \Theta)) \rangle Z, \quad (B18) \\
R_z &= \langle J_1 R_c \sin^2 (\theta - (\Theta_c - \Theta)) \rangle Z. \quad (B19)
\end{align*}

By transforming $\theta$ into $\theta' = \theta - (\Theta_c - \Theta)$ and since $A_{\theta' + \Theta_0 - \Theta}$ is periodic with period $\pi$ and an even function of $\theta'$, Eqs. (B17), (B18), and (B19) can be expressed as

\begin{align*}
R &= \frac{2}{\pi} \int_0^{\pi/2} d\psi' \int_0^\pi d\theta' g(\psi_0 + A_{\theta' + \Theta_0 - \Theta} \sin \psi) \\
& \times \cos^2 \psi (J_0 R + J_1 R_c \cos \theta'), \quad (B20) \\
R_c &= \frac{2}{\pi} \int_0^{\pi/2} d\psi' \int_0^\pi d\theta' g(\psi_0 + A_{\theta' + \Theta_0 - \Theta} \sin \psi) \\
& \times \cos^2 \psi (J_0 R + J_1 R_c \cos \theta') \cos \theta', \quad (B21) \\
R_z &= J_1 R_c \frac{2}{\pi} \int_0^{\pi/2} d\psi' \int_0^\pi d\theta' g(\psi_0 + A_{\theta' + \Theta_0 - \Theta} \sin \psi) \\
& \times \cos^2 \psi \sin \theta', \quad (B22)
\end{align*}

where $A_{\theta' + \Theta_0 - \Theta} = \sqrt{(J_0 R + J_1 R_c \cos \theta')^2 + (J_1 R_c \sin \theta')^2}$.

**APPENDIX C: CONDITION ON THE EXISTENCE OF THE $Pn$ SOLUTION, $J_1 > 2J_0$**

Let us consider a bifurcation from a solution with $R > 0$ and $R_1 = 0$ to a solution with $R > 0$ and $R_1 > 0$. Let us define $X(R)$ and $Y(R)$ as follows:

\begin{align*}
X(R) &\equiv \frac{2}{\pi} \int_0^{\pi/2} d\psi \, g(\omega_0 + J_0 R \sin \psi) \cos^2 \psi, \quad (C1) \\
Y(R) &\equiv -\frac{2}{\pi} \int_0^{\pi/2} d\psi \, g'(\omega_0 + J_0 R \sin \psi) \sin \psi \cos^2 \psi. \quad (C2)
\end{align*}

$X(R) > 0$ follows from Eq. (C1), since we assumed $R > 0$. Integrating the right-hand side of Eq. (C2) by parts yields

\begin{equation}
Y(R) = -\frac{2J_0 R}{3\pi} \int_0^{\pi/2} d\psi' \, g''(\omega_0 + J_0 R \sin \psi) \cos^4 \psi. \quad (C3)
\end{equation}

Here, we assume that $g''(x)$ exists. Note that $g'(\omega_0) = 0$ by definition. Since $\omega_0$ is the unique maximum of $g(\omega)$, $g''(\omega_0) \leq 0$ follows. Thus, $Y(R) > 0$ since $R > 0$. Defining $\varepsilon \equiv \frac{J_0 R}{3\pi R}$ for $\varepsilon \ll 1$, from Eqs. (B12) and (B13), we obtain

\begin{align*}
R &\simeq 2J_0 R \left( \frac{\pi}{2} - \frac{\pi}{8} J_0 R \varepsilon^2 Y \right), \quad (C4) \\
R_1 &\simeq 2J_1 R_1 \left( \frac{\pi}{4} - \frac{3\pi}{16} J_0 R \varepsilon^2 Y \right). \quad (C5)
\end{align*}

When $\varepsilon = 0$, $R_1 = 0$. Let us set $R = R^*$ at $\varepsilon = 0$. Then, from Eq. (C4), we obtain

\begin{equation}
1 = \pi J_0 R(X(R^*)), \quad (C6)
\end{equation}

Thus, $R^*$ satisfies Eq. (B1) for the $U$ solution. Furthermore, from Eq. (C5), we obtain

\begin{equation}
R_1 \simeq \frac{R^*(J_1 - 2J_0)}{3\pi J_1^2 Y(R^*)}. \quad (C7)
\end{equation}

Since $Y(R^*) > 0$, we find that the $Pn$ solution bifurcates from the $U$ solution at $J_1 = 2J_0$ and exists when $J_1 > 2J_0$.

**APPENDIX D: DERIVATIONS OF THE PHASE BOUNDARIES**

1. **Boundaries between the $S$ and $U$ phases**

We will analyze the SCES for the unstable $Pn$ solutions (B20), (B21), and (B22) and derive the boundary between the $S$ and $U$ phases. Assuming $R \ll 1$, we can expand $A_{\theta' + \Theta_0 - \Theta}$ into a Taylor series up to $O(R)$,

\begin{equation}
J_1 \tilde{A}(\theta') \equiv A_{\theta' + \Theta_0 - \Theta} \simeq J_1 \tilde{A}_0(\theta') + J_0 R_c \cos \theta' \tilde{A}_0(\theta'), \quad (D1)
\end{equation}

\begin{equation}
\tilde{A}_0(\theta') = \sqrt{(R_c \cos \theta')^2 + (R_z \sin \theta')^2}. \quad (D2)
\end{equation}

Thus, we obtain

\begin{align*}
g[\omega_0 + J_1 \tilde{A}(\theta') \sin \psi] \simeq g[\omega_0 + J_1 \tilde{A}_0(\theta') \sin \psi] \\
& + g'[\omega_0 + J_1 \tilde{A}_0(\theta') \sin \psi] \\
& \times J_0 R_c \cos \theta' \tilde{A}_0(\theta') \sin \psi.
\end{align*}

The SCES become

\begin{align*}
R &\simeq J_0 R Z_0 + J_1 R_c (\cos \theta') Z, \quad (D1) \\
R_c &\simeq J_0 R_c \cos (\theta') Z_0 + J_1 R_c (\cos \theta') Z, \quad (D2) \\
R_z &\simeq J_1 R_c \sin (\theta') Z, \quad (D3)
\end{align*}

\begin{equation}
\langle B \rangle_0 \equiv \frac{1}{2} \frac{2}{Z_0} \int_0^\pi d\psi \cos^2 \psi \\
& \times \int_0^\pi d\theta' g[\omega_0 + J_1 \tilde{A}_0(\theta') \sin \psi] B, \quad (D4)
\end{equation}
\[ Z_0 = \frac{2}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \times \int_0^{\pi} d\theta' g[\omega_0 + J_1 \hat{A}_0(\theta') \sin \psi]. \] (D5)

This can be rewritten as
\[ \hat{g}(x) = \sigma^2 g'(\omega_0 + \sigma x), \] and consequently \[ \hat{g}'(x) = \sigma^2 g'(\omega_0 + \sigma x) \] follows. Defining \( \tilde{J}_1 = \frac{J_1}{\sigma} \), we can rewrite Eqs. (D8) and (D10) as
\[ R_c = \tilde{J}_1 R_c \int_0^{\pi/2} d\psi \cos^2 \psi \hat{g}(\tilde{J}_1 R_c \sin \psi), \] (D11)
\[ \tilde{J}_0 = \left[ \frac{2}{\tilde{J}_1} + \tilde{J}_1 R_c \int_0^{\pi/2} d\psi \hat{g}'(\tilde{J}_1 R_c \sin \psi) \cos^2 \psi \sin \psi \right]^{-1}. \] (D12)

If \( g(\omega) \) is a Gaussian distribution with mean \( \omega_0 \) and standard deviation \( \sigma \), we have
\[ \hat{g}(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-x^2/\sigma^2}, \] \[ \hat{g}'(x) = -x \frac{1}{\sqrt{2\pi} \sigma} e^{-x^2/\sigma^2}. \]

## 2. Boundary between \( S \) and \( Pn \) phases

The boundary between \( S \) and \( Pn \) phases is obtained from the SCEs for the unstable \( Pn \) solution by setting \( R_c = 0 \). Assuming \( R_c \ll 1 \), we can expand \( \hat{A}(\theta') \) into a Taylor series up to \( O(R_c) \). Defining \( \hat{A}(\theta') = \hat{A}_0(\theta') + \sigma \), we obtain
\[ \hat{A}(\theta') = \sqrt{\hat{J}_0 R_c + \left( J_1 R_c \sin \theta' \right)^2} \]
\[ \approx \hat{A}_0(\theta') + \hat{J}_0 \hat{J}_1 R_c \cos \theta' \] \[ \approx \hat{A}_0(\theta') + \frac{\hat{J}_0 \hat{J}_1 R_c \cos \theta'}{\hat{A}_0(\theta')}, \]
\[ \hat{J}_0 \hat{J}_1 R_c \sin \theta'. \]

Accordingly, we have
\[ g[\omega_0 + \hat{J}_1 \hat{A}_0(\theta') \sin \psi] \approx g[\omega_0 + \sigma \hat{A}_0(\theta') \sin \psi] \]
\[ + \hat{g}[\hat{A}_0(\theta') \sin \psi] \left( \frac{\hat{J}_0 \hat{J}_1 R_c}{\hat{A}_0(\theta')} \sigma \sin \psi \right) \]
\[ \cos^2 \theta'. \]

The SCE (B21) for \( R_c \) becomes
\[ R_c \approx \frac{2}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \int_0^\pi d\theta' \left( \hat{g}[\hat{A}_0(\theta') \sin \psi] J_1 R_c \right. \]
\[ \left. + \hat{g}'[\hat{A}_0(\theta') \sin \psi] \left( \frac{\hat{J}_0 \hat{J}_1 R_c}{\hat{A}_0(\theta')} \sigma \sin \psi \right) \cos^2 \theta' \right. \]
\[ \left. \times \hat{A}_0(\theta') \right] \]
\[ \cos^2 \theta'. \] (D13)

Thus, the boundary between the \( S \) and \( Pn \) phases is given by
\[ 1 = \frac{2}{\pi} \hat{J}_1 \int_0^{\pi/2} d\psi \cos^2 \psi \int_0^\pi d\theta' \left( \hat{g}[\hat{A}_0(\theta') \sin \psi] \right. \]
\[ \left. + \hat{g}'[\hat{A}_0(\theta') \sin \psi] \left( \frac{\hat{J}_0 \hat{J}_1 R_c}{\hat{A}_0(\theta')} \sigma \sin \psi \right) \cos^2 \theta' \right. \]
\[ \left. \times \hat{A}_0(\theta') \right] \]
\[ \cos^2 \theta'. \] (D14)

On the other hand, on the boundary, the SCEs (B20) and (B22) for \( R \) and \( R_c \) become
\[ R = \sigma \hat{J}_0 R_c \frac{2}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \int_0^\pi d\theta' g[\omega_0 + \sigma \hat{A}_0(\theta') \sin \psi], \] (D15)
\[ R_c = \sigma \hat{J}_1 R_c \frac{2}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \]
\[ \times \int_0^\pi d\theta' g[\omega_0 + \sigma \hat{A}_0(\theta') \sin \psi] \sin^2 \theta'. \] (D16)
Since $R_c = 0$ on the boundary, $R_1 = R_c$ holds. Furthermore, since the integrand is symmetric with respect to $\theta' = \frac{\pi}{2}$, changing the integral range from $[0, \frac{\pi}{2}]$ to $[0, \frac{\pi}{2}]$ and making a variable transformation $\theta' \rightarrow \frac{\pi}{2} - \theta''$ yields

$$R = \sigma J_0 R \frac{4}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \times \int_0^{\pi/2} d\theta'' \frac{\sigma A_0(\theta'') \sin \psi}{}, \quad \text{(D17)}$$

$$R_1 = \sigma J_1 R \frac{4}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \times \int_0^{\pi/2} d\theta'' \frac{\sigma A_0(\theta'') \sin \psi \cos^2 \theta''}{}, \quad \text{(D18)}$$

$$\tilde{A}_0(\theta'') = \tilde{A}_0 \left( \frac{\pi}{2} - \theta'' \right) = \sqrt{(\tilde{J}_0 R)^2 + (\tilde{J}_1 R \cos \theta'')^2}.$$  

$$\text{(D19)}$$

These are merely the SCEs (B12), (B13), and (B14) for the stable $Pn$ solution. Now, let us summarize the formulas of the boundary between the $S$ and $Pn$ solutions,

$$J_1 = \left[ \frac{4}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \int_0^{\pi/2} d\theta'' \left( \tilde{g}_0(\theta'') \sin \psi \right) \tilde{A}_0(\theta'') \sin \psi \right]^{-1},$$

$$R = \tilde{J}_0 R \frac{4}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \times \int_0^{\pi/2} d\theta'' \tilde{g}_0(\theta'') \sin \psi,$$

$$R_1 = \tilde{J}_1 R \frac{4}{\pi} \int_0^{\pi/2} d\psi \cos^2 \psi \times \int_0^{\pi/2} d\theta'' \tilde{g}_0(\theta'') \sin \psi \cos^2 \theta''.$$  

$$\text{(D21)}$$

$$\tilde{A}_0(\theta'') = \sqrt{(\tilde{J}_0 R)^2 + (\tilde{J}_1 R \cos \theta'')^2}.$$  

$$\text{(D22)}$$

APPENDIX E: DERIVATION OF THE SCEs FOR THE SPINNING SOLUTION AND PENDULUM SOLUTION FOR MODEL 2

In model 2, the interaction is given by

$$J_{\theta \theta'} = \frac{1}{N} \left[ J_m \cos \{ m(\theta - \theta') \} + J_n \cos \{ n(\theta - \theta') \} \right]. \quad \text{(E1)}$$

where $m$ and $n$ are positive integers, and we assume $m < n$. Order parameters are defined as

$$R_{kc} e^{i\Phi_k} = \frac{1}{N} \sum_\theta \cos(k\theta) e^{i\phi_k}, \quad \text{(E2)}$$

$$R_{ks} e^{i\phi_k} = \frac{1}{N} \sum_\theta \sin(k\theta) e^{i\phi_k}. \quad \text{(E3)}$$

$k$ is $m$ or $n$. As usual, we assume that $R_{kc}$ and $R_{ks}$ tend to be constant and $\Theta_{kc} \rightarrow \omega_0 t + \Theta_{kc}', \Theta_{ks} \rightarrow \omega_0 t + \Theta_{ks}'$ as $t$ tends to $\infty$, and $\Theta_{kc}'$ and $\Theta_{ks}'$ are constant. The evolution equation for $\phi_0$ reduces to

$$\frac{d}{dt} \phi_0 = \omega_0 - A_{\theta} \sin(\phi_0 - \omega_0 t - \phi_0), \quad \text{(E4)}$$

$$A_{\theta} e^{i\psi_0} = \sum_{k=m,n} J_k [ R_{kc} \cos(k\theta) e^{i\phi_k} + R_{ks} \sin(k\theta) e^{i\phi_k} ].$$  

Defining $\psi_0 = \phi_0 - \omega_0 t - \phi_0$, the evolution equation becomes

$$\frac{d}{dt} \psi_0 = \omega_0 - A_{\theta} \sin \psi_0.$$  

$$\text{(E5)}$$

The order parameters are calculated as follows:

$$R_{kc} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} d\psi \frac{n_r(\theta, \psi) \cos(k\theta) e^{i(\psi + \phi_0 - \phi_0)}}{\text{cos}^2 \psi},$$

$$\text{(E6)}$$

$$R_{ks} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} d\psi \frac{n_r(\theta, \psi) \sin(k\theta) e^{i(\psi + \phi_0 - \phi_0)}}{\text{cos}^2 \psi}. \quad \text{(E7)}$$

From these equations, we obtain

$$R_{kc} = Z \sum_{k=m,n} J_k [ R_{kc} \cos(k\theta) \cos(k\theta) e^{i(\psi_0 - \psi_0)} + R_{ks} \sin(k\theta) \cos(k\theta) e^{i(\psi_0 - \psi_0)}],$$

$$\text{(E8)}$$

$$R_{ks} = Z \sum_{k=m,n} J_k [ R_{kc} \cos(k\theta) \sin(k\theta) e^{i(\psi_0 - \psi_0)} + R_{ks} \sin(k\theta) \sin(k\theta) e^{i(\psi_0 - \psi_0)}].$$

$$\text{(E9)}$$

$Z$ and $\langle \cdot \rangle$ mean the same as before. That is,

$$\langle B \rangle = \frac{1}{Z} \frac{1}{\pi} \int_0^{\pi/2} d\psi \int_0^{2\pi} d\theta \frac{g(\omega_0 + A_{\theta} \sin \psi) \cos^2 \psi B}{}, \quad \text{(E10)}$$

$$Z = \frac{1}{\pi} \int_0^{\pi/2} d\psi \int_0^{2\pi} d\theta \frac{g(\omega_0 + A_{\theta} \sin \psi) \cos^2 \psi}{}, \quad \text{(E11)}$$

We further define $R_k$ and $\phi_k$ by $R_k \rightarrow i R_k e^{i\phi_k}$. For simplicity, we omit primes from the $\Theta$’s except for the expressions of $\alpha_0$ and $\phi_0$.

1. Spinning solution

First, let us study the spinning solution. We will assume $R_m > 0, R_n = 0$ and examine the auxiliary equations, which are the imaginary parts of Eqs. (E8) and (E9),

$$R_m \sin(m \phi_m) (\cos(m \theta) \sin(m \theta)) \sin(\Theta_{mc} - \Theta_{ms}) = 0,$$

$$\text{(E12)}$$

$$R_m \cos(m \phi_m) (\cos(m \theta) \sin(m \theta)) \sin(\Theta_{mc} - \Theta_{ms}) = 0.$$  

$$\text{(E13)}$$

These equations are similar to Eqs. (A16) and (A17) in Appendix A. Therefore, for the stable $S$ solution, $\cos(\Theta_{mc} - \Theta_{ms}) = 0$ and $\langle \cos(m \theta) \rangle = 0$ follow, and
The SCEs are derived from the real parts of Eqs. (E8) and (E9).

The sufficient conditions for these equations are that the coefficients of $R_m$ and $R_s$ are 0. Accordingly, we obtain

$$\sin(m\phi_m)(\cos(m\theta)\sin(m\theta))\sin(\Theta_{mc} - \Theta_{ms}) = 0, \quad (E26)$$

$$J_n R_n (n\phi_n)\cos(n\theta)\sin(n\theta))\sin(\Theta_{nc} - \Theta_{ns}) = 0. \quad (E27)$$

For the pendulum solution, we assume $\sin(\Theta_{nc} - \Theta_{ns}) = 0$ and $\sin(\Theta_{nc} - \Theta_{ms}) = 0$ because similar conditions are obtained in model 1 for the stable $P_n$ solution.

[The imaginary part of Eq. (E8) with $k = m + i\times$ the imaginary part of Eq. (E9) with $k = m$, and [the imaginary part of Eq. (E8) with $k = n + i\times$ the imaginary part of Eq. (E9) with $k = n$] reduce to

$$\sin(\Theta_{nc} - \Theta_{ms})(\phi_{nc} - \phi_{ns})\cos(n\theta' - \tilde{\theta}) = 0, \quad (E28)$$

$$\sin(\Theta_{nc} - \Theta_{ms})\sin(n\theta' - \tilde{\theta}) = 0. \quad (E29)$$

Here, we define

$$\tilde{\theta}_h = \phi_{kc} e^{i(\Theta_{rc} - \Theta_{rc})}, \quad \tilde{\theta}_k = \phi_{kc} (\Theta_{kc} - \Theta_{kc}),$$

$$\tilde{\theta} = \tilde{\theta}_h - \tilde{\theta}_m, \quad \theta' = \tilde{\theta} - \tilde{\theta}_m.$$

We assume that $\sin(\Theta_{nc} - \Theta_{ns}) \neq 0$. From the real and imaginary parts of Eq. (E28), we obtain

$$\langle \cos(m\theta)\cos(n\theta' - \tilde{\theta}) \rangle = 0, \quad (E30)$$

$$\langle \sin(m\theta)\cos(n\theta' - \tilde{\theta}) \rangle = 0. \quad (E31)$$

From the real and imaginary parts of Eq. (E29), we obtain

$$\langle \phi_{nc} - \phi_{ns} \rangle = 0, \quad (E32)$$

$$\langle \mathcal{C}(m\theta) \rangle = 0, \quad (E33)$$

[Equation (E30) + $i\times$ Eq. (E31)] and [Eq. (E32) + $i\times$ Eq. (E33)] reduce to

$$\langle e^{i\theta'\phi_{nc} - \phi_{ns}} \rangle = 0, \quad (E34)$$

$$\langle e^{i\theta'\phi_{nc} - \phi_{ns}} \rangle = 0. \quad (E35)$$

where $\theta$ has been replaced with $\theta' + \tilde{\theta}_m$. From Eq. (E35), we obtain

$$\langle \cos((m + n)\theta') + \cos((m - n)\theta') \rangle = 0, \quad (E36)$$

$$\langle \sin((m + n)\theta') + \sin((m - n)\theta') \rangle = 0. \quad (E37)$$

On the other hand, by using Eq. (E35), Eq. (E34) reduces to

$$\langle \sin(n\theta' - \tilde{\theta}) \rangle = 0. \quad (E38)$$

$A_{\theta}$ is expressed as

$$A_{\theta} = A_{\theta' + \tilde{\theta}_m},$$

where $\theta'$ has been replaced with $\theta' + \tilde{\theta}_m$. From Eq. (E35), we obtain

$$\langle \cos((m + n)\theta') + \cos((m - n)\theta') \rangle = 0, \quad (E36)$$

$$\langle \sin((m + n)\theta') + \sin((m - n)\theta') \rangle = 0. \quad (E37)$$

On the other hand, by using Eq. (E35), Eq. (E34) reduces to

$$\langle \sin(n\theta' - \tilde{\theta}) \rangle = 0. \quad (E38)$$

$A_{\theta}$ is expressed as

$$A_{\theta} = A_{\theta' + \tilde{\theta}_m}.$$
Therefore, we obtain
\[ A_{\varphi+\delta_n} = \sqrt{[J_m R_m \cos(m \theta')]^2 + [J_n R_n \cos(n(\theta' - \vartheta))]^2}.\]  
(E41)

Therefore, Eqs. (E36) and (E37) reduce to
\[ \langle \cos((n-m)\theta') \rangle = 0, \]
\[ \langle \sin((n-m)\theta') \rangle = 0. \]  
(E42)

Therefore, Eq. (E38) is satisfied. If \( n \neq 3m \), conditions (E42) and (E43) are automatically satisfied. For simplicity, we will assume \( n \neq 3m \). Thus, the conditions for \( P_n \) are
\[ \sin(\Theta_{mc} - \Theta_{ns}) = 0, \quad \sin(\Theta_{nc} - \Theta_{ns}) = 0, \]
\[ \cos(\Theta_{nc} - \Theta_{ms}) = 0. \]  
(E44)

Now, let us study the SCEs, which are the real parts of Eqs. (E8) and (E9). [The real part of Eq. (E8) with \( k = m \) + \([i \times \text{the real part of Eq. (E9)}] \) \( k = m \)] and [the real part of Eq. (E8) with \( k = n \) + \([i \times \text{the real part of Eq. (E9)}] \) \( k = n \)] give the SCEs,
\[ 1 = J_m Z \left(1 + e^{2i m \theta' \cos(\Theta_{mc} - \Theta_{ns})} \right), \]
\[ 1 = J_n Z \left(1 + e^{2i n \theta' \cos(\Theta_{nc} - \Theta_{ns})} \right). \]  
(E45)

The real and imaginary parts of Eq. (E45) are
\[ 1 = J_m Z \langle \cos^2(m \theta') \rangle, \]
\[ \langle \sin(2m \theta') \rangle = 0. \]  
(E46)

The real and imaginary parts of Eq. (E46) are
\[ 1 = J_n Z \langle \cos^2(n \theta' - \vartheta) \rangle, \]
\[ \langle \sin(2n \theta' - \vartheta) \rangle = 0. \]  
(E50)

Equation (E48) is automatically satisfied since \( \bar{A}_\theta \) does not contain the factor \( \sin(2m \theta') \). From Eq. (E50), we have
\[ \langle \sin(2m \theta') \rangle \cos(2m \vartheta) = \langle \cos(2m \vartheta) \rangle \sin(2m \vartheta). \]  
(E51)

The sufficient condition for this is \( \cos(2m \vartheta) = 0 \) or \( \sin(2m \vartheta) = 0 \), and it determines the value of \( \vartheta \).

Therefore, the SPEs are
\[ 1 = J_n Z \langle \cos^2(n \theta' - \vartheta) \rangle, \]
\[ 1 = J_n Z \langle \cos^2(n \theta' - \vartheta) \rangle. \]  
(E53)

By changing the variable from \( \theta \) to \( \theta' \), the SPEs can be expressed as
\[ 1 = \frac{J_m}{\pi} \int_0^{\pi/2} d\psi \int_0^{2\pi} d\theta' g(\omega_0 + A_{\varphi+\delta_n} \sin \psi) \times \cos^2 \psi \cos^2(m \theta'), \]
\[ 1 = \frac{J_n}{\pi} \int_0^{\pi/2} d\psi \int_0^{2\pi} d\theta' g(\omega_0 + A_{\varphi+\delta_n} \sin \psi) \times \cos^2 \psi \cos^2(n \theta' - \vartheta), \]  
(E54)

where \( A_{\varphi+\delta_n} = \sqrt{[J_m R_m \cos(m \theta')]^2 + [J_n R_n \cos(n(\theta' - \vartheta))]^2}. \)

Setting \( \bar{\alpha}_n = A_0 - \Theta_{ms} \), we have
\[ A_0 \cos \bar{\alpha}_n = J_m R_m \cos(\Theta_{mc} - \Theta_{ms}) \cos(m \theta'), \]
\[ A_0 \sin \bar{\alpha}_n = J_n R_n \sin(\Theta_{nc} - \Theta_{ms}) \cos(n \theta' - \vartheta). \]  
(E56)

Thus,
\[ \phi^n_0 = \omega_0 t + \sin^{-1} \left( \frac{\omega_0 - \omega_n}{A_0} \right) + \bar{\alpha}_n + \Theta_{ms}. \]  
(E58)

This formula shows that the rotation number can take on values of 0, \( \pm 1, \ldots, \pm m \).
