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<tr>
<td>著者</td>
<td>高須 夫悟</td>
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<td>キーワード</td>
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URL: [http://nwudir.lib.nara-w.ac.jp/dspace](http://nwudir.lib.nara-w.ac.jp/dspace)
1 Probability generating function

The population size \( n \) is a discrete random number and is associated with a probability distribution \( P_n(t) \) which evolves with time \( t \) according to master equation. Solving \( P_n(t) \) in general is not always easy but for some simple case we can do it. We now introduce **probability generating function** \( G(t, z) \) associated with a probability distribution \( P_n(t) \) as

\[
G(t, z) = \sum_n P_n(t) z^n \tag{1}
\]

where the summation is taken for all possible \( n \). This series will converge when \( |z| < r \) where \( r \) is convergence radius. Probability generating function is just a power series whose \( i \)-th coefficient is \( P_i(t) \).

Let’s take a look of general properties of probability generating function. It is obvious that \( G(t, z) \) evaluated at \( z = 1 \) is always 1 because it is just summation of probability \( P_n(t) \).

\[
G(t, 1) = \sum_n P_n(t) = 1 \tag{2}
\]

Differentiating (2) with \( z \) yields

\[
\frac{\partial}{\partial z} G(t, z) = \sum_n n P_n(t) z^{n-1}
\]

and by substituting \( z = 1 \) we have a useful result

\[
\frac{\partial}{\partial z} G(t, z) \bigg|_{z=1} = \sum_n n P_n(t) = \langle n \rangle = E[n] \tag{3}
\]

Similarly differentiating (2) with \( z \) twice

\[
\frac{\partial^2}{\partial z^2} G(t, z) = \sum_n n(n-1) P_n(t) z^{n-2}
\]
and substituting $z = 1$ gives
\[
\frac{\partial^2 G(t, z)}{\partial z^2} \bigg|_{z=1} = \sum_n n(n-1)P_n(t) = E[n(n-1)] = \langle n^2 \rangle - \langle n \rangle
\] (4)

We now remember that $\text{Var}[n] = \langle n^2 \rangle - \langle n \rangle^2$. That is,
\[
\text{Var}[n] = \left\{ \frac{\partial^2 G(t, z)}{\partial z^2} + \frac{\partial}{\partial z} G(t, z) - \left( \frac{\partial}{\partial z} G(t, z) \right)^2 \right\} \bigg|_{z=1}
\] (5)

These calculations show that if we can solve and obtain a probability generating function $G(t, z)$, we can derive the ensemble average $\langle n \rangle$ and the variance $\text{Var}[n]$ from $G(t, z)$. In previous lectures, we derived moment dynamics directly from master equation. But they can be also obtained from $G(t, z)$. Even more, probability $P_n(t)$ is given as a coefficient of Taylor expansion of $G(t, z)$ around $z = 0$. This means solving $G(t, z)$ is equivalent to solving $P_n(t)$. In the following sections we try to solve the p.g.f. $G(z, t)$ of the stochastic immigration-emigration process.

2 Solving the pgf of immigration-emigration process

We now solve the pgf of immigration-emigration process in which $n$ can be negative (population size is no longer restricted non-negative). The master equation is
\[
\frac{dP_n(t)}{dt} = \alpha P_{n-1}(t) + \beta P_{n+1}(t) - (\alpha + \beta)P_n(t) \quad \text{for } -\infty < n < \infty
\] (6)

and the pgf is defined as
\[
G(t, z) = \sum_n P_n(t)z^n
\] (7)

Differentiating the pgf with $t$ yields
\[
\frac{\partial}{\partial t} G(t, z) = \sum_n \frac{d}{dt} P_n(t)z^n
\]

Using the master equation (6), we have
\[
\frac{\partial}{\partial t} G(t, z) = \sum_n \left\{ \alpha P_{n-1}(t) + \beta P_{n+1}(t) - (\alpha + \beta)P_n(t) \right\} z^n
\]
\[
= \alpha z \sum_n P_{n-1}(t)z^{n-1} + \beta z \sum_n P_{n+1}(t)z^{n+1} - (\alpha + \beta) \sum_n P_n(t)z^n
\]
\[
= (\alpha z + \beta / z - \alpha - \beta)G(t, z)
\]

This is ODE of $G(t, z)$ with respect to time $t$ and has unique solution with initial condition $G(0, z) = z^m$ where $m$ is initial population size at $t = 0$, $n(0)$. This can be readily solved by variable separation. The solution is
\[
G(t, z) = z^m \exp \left[ (-\alpha - \beta + \alpha z + \beta / z)t \right]
\] (8)
3 Problem

We have solved the pgf of the immigration-emigration process $G(t, z)$.

1. Confirm that the expected value $E[n]$ and variance $Var[n]$ of $n$ derived from the moment dynamics in the last lecture coincide with those derived from the pgf.

2. By Taylor expanding the pgf $G(t, z)$ with respect to $z$ and looking at coefficients of $z^n$, $P_n(t)$ can be obtained. This is not actually easy but we are a bit close to the solution $P_n(t)$. 