<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>Exponent of inverse local time for harmonic transformed process</td>
</tr>
<tr>
<td>著者</td>
<td>Takemura Tomoko; Tomisaki Matsuyo</td>
</tr>
<tr>
<td>発行</td>
<td>人間文化研究科年報（奈良女子大学大学院人間文化研究科）、第31号、pp. 127-138</td>
</tr>
<tr>
<td>発行日</td>
<td>2016-03-31</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10935/4204">http://hdl.handle.net/10935/4204</a></td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
</tr>
</tbody>
</table>

https://nwudir.lib.nara-w.ac.jp/dspace
Exponent of inverse local time for harmonic transformed process

TAKEMURA Tomoko* and TOMISAKI Matsuyo†

1 Introduction

In this paper we are concerned with inverse local time at the regular end point for harmonic transform of one dimensional diffusion process, and consider the corresponding exponent, entrance law and excursion law. On the exponent of inverse local time, there are some interesting works due to R. M. Blumenthal and R. K. Getoor [1], M. Fukushima and H. Tanaka [3], K. Itô and H. P. McKeans [4], etc. In [4], K. Itô and H. P. McKeans showed that the Lévy measure density corresponding to the inverse local time at the regular end point for a recurrent one dimensional diffusion process is represented as the Laplace transform of the spectral measure corresponding to the diffusion process, where the absorbing boundary condition is posed at the end point. In our previous paper [9], employing their representation theorem, we showed that the Lévy measure density corresponding to the inverse local time at the regular end point for a recurrent harmonic transformed process is represented as the Laplace transform of the spectral measure corresponding to the original diffusion process, where the absorbing boundary condition is posed at the end point if it is regular. In this paper, we show that a representation theorem due to K. Itô and H. P. McKeans is available for a transient one dimensional diffusion process, and deduce a representation theorem of the Lévy measure density corresponding to the inverse local time for a transient harmonic transformed process. Furthermore we show a relation between exponents of inverse local time in [1] and [3], and relations between entrance laws or excursion laws of original diffusion process and its harmonic transform. So far we have treated harmonic transform of minimal processes. However it is possible to consider harmonic transform of non-minimal processes. We present a new consideration for harmonic transform of non-minimal processes.

2 One dimensional diffusion processes

Let $s$ be a continuous increasing function on an open interval $I = (l_1, l_2)$, where $-\infty \leq l_1 < l_2 \leq \infty$, $m$ be a right continuous increasing function on $I$ and $k$ be a right continuous nondecreasing function on $I$. We sometimes use the same symbols $s$, $m$ and $k$ for the induced measures $ds(x)$, $dm(x)$ and $dk(x)$, respectively. For a function $u$ on $I$, we set $u(l_i) = \lim_{x \to l_i, x \in I} u(x)$ if there exists the limit, for $i = 1, 2$. We denote by $D_s u(x)$ the right derivative with respect to $s(x)$, that is, $D_s u(x) = \lim_{\varepsilon \downarrow 0} \{u(x + \varepsilon) - u(x)\}/\{s(x + \varepsilon) - s(x)\}$, provided it exists. Let us fix a point $c_0 \in I$ arbitrarily and set $J_{\mu,\nu}(x) = \int_{[c_0,x]} d\mu(y) \int_{[c_0,y]} d\nu(z)$, for $x \in I$, where $\mu$ and $\nu$ are Borel measures on $I$, and the

* Faculty(Natural Sciences, Mathematics), Assistant Professor
† Nara Women’s University, Professor Emeritus
integral \( \int_{(a,b]} \) is read as \(- \int_{(b,a]} \) if \( a > b \). Following [2], we call the boundary \( l_i \) to be

- (s, m, k)-regular if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- (s, m, k)-exit if \( J_{s,m+k}(l_i) < \infty \) and \( J_{m+k,s}(l_i) = \infty \),
- (s, m, k)-entrance if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) < \infty \),
- (s, m, k)-natural if \( J_{s,m+k}(l_i) = \infty \) and \( J_{m+k,s}(l_i) = \infty \).

If \( l_i \) is \((s, m, k)\)-regular, then the following boundary condition is posed at \( l_i \) for each \( i = 1, 2 \).

\[
(2.1) \quad a_i u(l_i) + (-1)^i b_i D_s u(l_i) = 0,
\]

where \( a_i \) and \( b_i \) are nonnegative real numbers satisfying \( a_i + b_i > 0 \). Here the above boundary condition with \( a_i > 0 \) and \( b_i = 0 \), \( a_i = 0 \) and \( b_i > 0 \), \( a_i > 0 \) and \( b_i = 0 \) is called absorbing, reflecting, elastic, respectively.

Let \( \mathcal{G}_{s,m,k} \) be a one dimensional generalized diffusion operator on \( I \) with a scale function \( s \), a speed measure \( m \), and a killing measure \( k \). We denote by \( \mathbb{D}_{s,m,k} = [X(t), P_x] \) the one dimensional diffusion process on \( I \) whose generator is given by \( \mathcal{G}_{s,m,k} \).

For \( \alpha \geq 0 \) and \( i = 1, 2 \), let \( g_i(\cdot, \alpha) \) be a function on \( I \) satisfying the following properties (2.2) through (2.6).

\[
(2.2) \quad g_i(x, \alpha) \text{ is positive and continuous in } x.
\]

\[
(2.3) \quad g_i(x, \alpha) \text{ is nondecreasing in } x.
\]

\[
(2.4) \quad g_2(x, \alpha) \text{ is nonincreasing in } x.
\]

\[
(2.5) \quad a_i g_i(l_i, \alpha) + (-1)^i b_i D_s g_i(l_i, \alpha) = 0
\]

if \( l_i \) is \((s, m, k)\)-regular and (2.1) is posed,

\[
(2.6) \quad g_i(x, \alpha) = g_i(c_o, \alpha) + D_s g_i(c_o, \alpha) \{ s(x) - s(c_o) \}
\]

\[
+ \int_{[c_o,x]} \{ s(x) - s(y) \} g_i(y, \alpha) \{ \alpha dm(y) + dk(y) \}, \quad x \in I.
\]

It is known that there exist functions \( g_i(\cdot, \alpha), \quad i = 1, 2 \), satisfying the properties (2.2) through (2.6) (see [4]).

Now we set \( W(\alpha) = D_s g_1(x, \alpha) g_2(x, \alpha) - g_1(x, \alpha) D_s g_2(x, \alpha) \). Note that \( W(\alpha) \) is a positive number independent of \( x \in I \). We put

\[
G(\alpha, x, y) = G(\alpha, y, x) = W(\alpha)^{-1} g_1(x, \alpha) g_2(y, \alpha),
\]

for \( \alpha > 0, \quad x, y \in I, \quad x \leq y \). \( G(\alpha, x, y) \) is the \( \alpha \)-Green function corresponding to the \( \mathbb{D}_{s,m,k} \). It is known that there exists a positive continuous function \( p(t, x, y) \) such that

\[
G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) dt, \quad \alpha > 0, \quad x, y \in I.
\]

It is easy to see that, if \( k \neq 0 \), then there exists \( G(0, x, y) \). In case \( k = 0 \), there exists \( G(0, x, y) \) if and only if \( |s(l_i)| < \infty \) for \( i = 1 \) or 2 (see [11]). We note that \( p(t, x, y) \) is the transition probability density with respect to \( dm \) for \( \mathbb{D}_{s,m,k} \), that is,

\[
P_x(X(t) \in E) = \int_E p(t, x, y) dm(y), \quad t > 0, \quad x \in I, \quad E \in \mathcal{B}(I),
\]

where \( \mathcal{B}(I) \) stands for the set of all Borel sets of \( I \).
Now we denote by $\mathbb{D}^{0}_{s,m,k} = [X(t), P_{x}^{0}]$ [resp. $\mathbb{D}^{*}_{s,m,k} = [X(t), P_{x}^{*}]$] the one dimensional diffusion process on $I$ whose generator is given by $\mathcal{G}_{s,m,k}$ when $l_{1}$ is $(s, m, k)$-regular with $a_{1} > 0$ and $b_{1} = 0$, i.e. $l_{1}$ is absorbing [resp. $a_{1} = 0$ and $b_{1} > 0$, i.e. $l_{1}$ is reflecting]. Let denote by $p^{0}(t, x, y)$ [resp. $p^{*}(t, x, y)$] the transition probability density with respect to $dm$ for $\mathbb{D}^{0}_{s,m,k}$ [resp. $\mathbb{D}^{*}_{s,m,k}$].

It is well known that $p^{0}(t, x, y)$ and $p^{*}(t, x, y)$ are represented as

\[(2.7)\quad p^{0}(t, x, y) = \int_{(0, \infty)} e^{-\lambda t} \psi^{0}(x, -\lambda) \psi^{0}(y, -\lambda) d\sigma^{0}(\lambda),\]

\[(2.8)\quad p^{*}(t, x, y) = \int_{(0, \infty)} e^{-\lambda t} \psi^{*}(x, -\lambda) \psi^{*}(y, -\lambda) d\sigma^{*}(\lambda),\]

for $t > 0$, $x, y \in I$, where $d\sigma^{0}(\lambda)$ and $d\sigma^{*}(\lambda)$ are Borel measures on $[0, \infty)$ satisfying

\[(2.9)\quad \int_{(0, \infty)} e^{-\lambda t} d\mu(\lambda) < \infty, \quad t > 0, \quad \mu = \sigma^{0}, \sigma^{*},\]

and $\psi^{0}(x, \alpha)$ and $\psi^{*}(x, \alpha)$, $x \in I$, $\alpha \in \mathbb{C}$, are the solutions of the following integral equations.

\[(2.10)\quad \psi^{0}(x, \alpha) = s(x) - s(l_{1}) + \int_{(l_{1}, x]} \{s(x) - s(y)\} \psi^{0}(y, \alpha) \{\alpha dm(y) + dk(y)\},\]

\[(2.11)\quad \psi^{*}(x, \alpha) = 1 + \int_{(l_{1}, x]} \{s(x) - s(y)\} \psi^{*}(y, \alpha) \{\alpha dm(y) + dk(y)\}.\]

Further, if $l_{2}$ is $(s, m, k)$-regular and the boundary condition (2.1) is posed at $l_{2}$, then $\psi^{0}(x, \alpha)$ and $\psi^{*}(x, \alpha)$ satisfy the condition (2.1).

In the case that $l_{1}$ is not $(s, m, k)$-regular, $p(t, x, y)$ is not always represented like (2.7) or (2.8) with $d\sigma(\lambda)$ satisfying (2.9) and $\psi^{0}(x, -\lambda)$ or $\psi^{*}(x, -\lambda)$. In [9] we gave a sufficient condition for $p(t, x, y)$ having a representation such as (2.8) in the case that $l_{1}$ is $(s, m, k)$-entrance.

3 Harmonic transform of $\mathbb{D}_{s,m,k}$

For $\beta \geq 0$, let $h_{\beta}(\cdot)$ be a positive continuous function on $I$ satisfying

\[(3.1)\quad h_{\beta}(x) = h_{\beta}(c_{o}) + D_{s} h_{\beta}(c_{o}) \{s(x) - s(c_{o})\} + \int_{(c_{o}, x]} \{s(x) - s(y)\} h_{\beta}(y) \{\beta dm(y) + dk(y)\}, \quad x \in I.\]

There exists such a function $h_{\beta}(\cdot)$. Indeed, it is represented as a linear combination of $g_{i}(\cdot, \beta)$, $i = 1, 2$. Note that $0 \leq h(l_{1}) \leq \infty$. However we assume $0 < h(l_{1}) < \infty$ whenever $l_{1}$ is $(s, m, k)$-regular, and reflecting or elastic. This assumption is natural following [14].

Let $\mathcal{H}_{s,m,k,\beta}$ be the set of all positive functions $h_{\beta}$ satisfying (3.1). For $h \in \mathcal{H}_{s,m,k,\beta}$, we set

\[p_{h}(t, x, y) = e^{-\beta t} \frac{p(t, x, y)}{h(x)h(y)}, \quad t > 0, \ x, y \in I,\]

\[G_{h}(\alpha, x, y) = \frac{G(\alpha + \beta, x, y)}{h(x)h(y)}, \quad \alpha > 0, \ x, y \in I.\]
We also set
\[ s_h(x) = \int_{(c_u, x]} h(y)^{-2} \, ds(y), \quad m_h(x) = \int_{(c_u, x]} h(y)^2 \, dm(y). \]

It is easy to see that \( p_{(h)}(t, x, y) \) is the transition probability density with respect to \( dm_h \) corresponding to a one dimensional diffusion process \( \mathbb{D}_{(h)} \), which is the harmonic transform of \( \mathbb{D}_{s,m,k} \) based on \( h \in \mathcal{H}_{s,m,k,\beta} \). Under the setting that the absorbing boundary condition is posed at \( l_i \) whenever it is regular, we proved that \( \mathbb{D}_{(h)} \) coincides with \( \mathbb{D}_{s,h,0} \), the one dimensional diffusion process on \( I \) with scale \( s_h \), speed measure \( m_h \), and null killing measure (see [12]). Further we studied how the state of boundaries is transposed (see [8]). In this paper we treat the boundary condition (2.1), and hence we have to note the following.

**Proposition 3.1** Assume that \( l_i \) is \((s, m, k)\)-regular and (2.1) is posed at \( l_i \), and \( 0 < h(l_i) < \infty \). Then \( l_i \) is \((s_h, m_h, 0)\)-regular and the following boundary condition is satisfied.

\[ a_{(h),i} u(l_i) + (-1)^i b_i D_{s_h}(l_i) = 0, \]

where \( a_{(h),i} = h(l_i)\{a_i h(l_i) + (-1)^i b_i D_h(l_i)\} \).

We get \( a_{(h),i} \geq 0 \) by using the fact that \( h \) is represented as a linear combination of \( g_i(\cdot, \beta) \), \( i = 1, 2 \). The above proposition is proved by the same method as in Lemma 3.3 of [12].

Combining Proposition 3.1 and some results in [8], we summarize the state of boundaries for \( \mathbb{D}_{(h)} \) as follows.

<table>
<thead>
<tr>
<th>State of Boundaries</th>
<th>( h(l_i) = 0 )</th>
<th>( h(l_i) \in (0, \infty) )</th>
<th>( h(l_i) = \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((s, m, k))-regular and absorbing</td>
<td>((s_h, m_h, 0))-regular and absorbing</td>
<td>((s_h, m_h, 0))-regular and absorbing</td>
<td>\emptyset</td>
</tr>
<tr>
<td>((s, m, k))-regular, and reflecting or elastic</td>
<td>\emptyset</td>
<td>((s_h, m_h, 0))-regular, and reflecting or elastic</td>
<td>\emptyset</td>
</tr>
<tr>
<td>((s, m, k))-exit</td>
<td>((s_h, m_h, 0))-exit</td>
<td>( (s_h, m_h, 0))-exit</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>
| \((s, m, k)\)-entrance | \emptyset | \((s_h, m_h, 0)\)-entrance | \( (s_h, m_h, 0)\)-regular and absorbing if \( |m_h(l_i)| < \infty \)
| \((s, m, k)\)-natural if \( J_{m_h, s_h}(l_i) < \infty \) | \( (s_h, m_h, 0)\)-natural if \( J_{m_h, s_h}(l_i) < \infty \) | \emptyset | \( (s_h, m_h, 0)\)-natural if \( |m_h(l_i)| = \infty \) and \( J_{m_h, s_h}(l_i) < \infty \)
| \( (s, m, k)\)-natural if \( J_{m_h, s_h}(l_i) = \infty \) | \emptyset | \( (s_h, m_h, 0)\)-natural if \( |m_h(l_i)| = \infty \) and \( J_{s_h, m_h}(l_i) = \infty \) | \emptyset |

The symbol \( \emptyset \) of the table means that there is no elements of \( \mathcal{H}_{s,m,k,\beta} \).

When \( l_i \) is \((s_h, m_h, 0)\)-regular for \( \mathbb{D}_{(h)} \), we can pose the absorbing boundary condition at \( l_i \). We denote by \( \mathbb{D}_{s_h, m_h, 0}^{(h)} \) such diffusion process on \( I \). We also denote by \( p_{(h)}^{(i)}(t, x, y) \) the transition probability density with respect to \( dm_h \) corresponding to \( \mathbb{D}_{(h)}^{(i)} \). In view of the uniqueness of the transition probability density, we obtain the following.
Proposition 3.2 Assume that $l_1$ is $(s_h, m_h, 0)$-regular for $\mathbb{D}((h))$. Then
\[ p^0_{(h)}(t, x, y) = e^{-\beta t} \tilde{p}(t, x, y) / h(x)h(y), \quad t > 0, \ x, y \in I, \]
where
\[ \tilde{p}(t, x, y) = \begin{cases} p^0(t, x, y), & \text{if } l_1 \text{ is } (s, m, k)-\text{regular and absorbing}, \\ p(t, x, y), & \text{if } l_1 \text{ is } (s, m, k)-\text{entrance or -natural}. \end{cases} \]

4 Inverse local time

Let $\mathbb{D}^0_{s,m,k} = [X(t), P^0_x]$ [resp. $\mathbb{D}^*_{s,m,k} = [X(t), P^*_x]$] be the one dimensional diffusion process on $I$ defined in Section 2.

Now we consider $\mathbb{D}^*_{s,m,k} = [X(t), P^*_x]$ and assume that the killing measure is null. We consider the local time $l(t, \xi)$, that is, $\int_0^t f(X(u)) \, du = \int_I l(t, \xi) f(\xi) \, d\mu(\xi), \quad t > 0$, for bounded continuous functions $f$ on $I$. Since $l(t, \xi)$ is continuous and nondecreasing in $t$ $P^*_x$-a.s., there is the right continuous inverse function $l^{-1}(t, \xi)$. In particular, we denote by $\tau^*(t)$ the inverse local time $l^{-1}(t, l_1)$ at the end point $l_1$.

The following result is obtained by K. Itô and H. P. McKean [4] in case $P^*_x(\sigma_{1_1} < \infty) = 1$, where $\sigma_a$ denotes the first hitting time for a point $a$.

Proposition 4.1 Assume that the killing measure is null, and $l_1$ is $(s, m, 0)$-regular and reflecting. Then there are a real number $\gamma^*$ and Lévy measure density $n^*(\xi)$ such that
\begin{equation}
E^*_{l_1} \left[ e^{-\lambda \gamma^*(t)} \right] = \exp \left\{ -\gamma^* t - t \int_0^\infty (1 - e^{-\lambda \xi}) n^*(\xi) \, d\xi \right\},
\end{equation}
\begin{equation}
\gamma^* = \begin{cases} \frac{a_2}{\{s(l_2) - s(l_1)\}} + b_2 & \text{if } l_2 \text{ is } (s, m, 0)-\text{regular and the boundary condition (2.1) is posed at } l_2, \\ 1/\{s(l_2) - s(l_1)\} & \text{otherwise}, \end{cases}
\end{equation}
\begin{equation}
n^*(\xi) = \lim_{x,y \to l_1} D_{s(x)}D_{s(y)} p^0(\xi, x, y) = \int_{[0, \infty)} e^{-\lambda \xi} d\sigma^0(\lambda),
\end{equation}
where $d\sigma^0(\lambda)$ is the Borel measure appeared in the representation (2.7) satisfying (2.9).

Here and hereafter we use the conventions $1/\infty = 0$ and $\pm a/0 = \pm \infty$ for $a > 0$.

Proof When $P^*_x(\sigma_{1_1} < \infty) = 1$, K. Itô and H. P. McKean showed (4.1) with $\gamma^* = 0$ and $n^*(\xi)$ given by (4.3) (see Section 6.2 of [4]). Note that $P^*_x(\sigma_{1_1} < \infty) = 1$ if and only if $s(l_2) = \infty$ or $l_2$ is $(s, m, 0)$-regular and reflecting, that is, $i = 2$ and $a_2 = 0$ in (2.1). Therefore the righthand side of (4.2) is equal to zero. Thus the proposition holds true in this case.

When $P^*_x(\sigma_{1_1} < \infty) < 1$, their argument still works well. We explain it. In this case that $s(l_2) < \infty$ and, in particular, if $l_2$ is $(s, m, 0)$-regular, then it is absorbing or elastic, that is, (2.1) is posed with $i = 2$ and $a_2 > 0$. We denote by $G^*(\alpha, x, y)$, $g^*_i(\alpha, x)$, etc. the items corresponding to $\mathbb{D}^*_{s,m,0}$. Since $l_1$ is $(s, m, 0)$-regular and reflecting, by the same method as for (1b) in Section 6.2 of [4], we get the following.
\begin{equation}
E^*_{l_1} \left[ e^{-\lambda \gamma^*(t)} \right] = \exp \left\{ -t/G^*(\lambda, l_1, l_1) \right\} = \exp \left\{ tD_{s(l_2)}g^*_2(l_1, \lambda)/g^*_2(l_1, \lambda) \right\}.
\end{equation}

By the same reason as for (4) in Section 6.2 of [4], there exists the limit
\begin{equation}
\tilde{n}(\xi) = \lim_{x \to l_1} \{P^*_x(\sigma_{1_1} \leq \xi) - 1\} / (s(x) - s(l_1)) \in [-\infty, 0),
\end{equation}
which is nondecreasing and right continuous. Since $g_2^*(x, \lambda)/g_2^*(l_1, \lambda) = E^*_{\lambda} [\exp \{-\lambda \sigma_1 \}]$, we get the following

$$
(4.5) \quad \frac{D_s g_2^*(l_1, \lambda)}{g_2^*(l_1, \lambda)} = \lim_{x \downarrow l_1} \frac{1}{s(x) - s(l_1)} \left\{ E^*_{\lambda} \left[e^{-\lambda \sigma_1} \right] - 1 \right\}
$$

$$
= \lim_{x \downarrow l_1} \frac{1}{s(x) - s(l_1)} \left\{ \int_{(0, \infty)} (e^{-\lambda \xi} - 1) P_1^*(\sigma_1 \in d\xi) - P_1^*(\sigma_1 = \infty) \right\}
$$

$$
= - \int_{0}^{\infty} \tilde{n}(\xi) d_\xi(e^{-\lambda \xi} - 1).
$$

We next note the following.

$$
(4.6) \quad \lim_{\xi \to 0} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) = 0,
$$

$$
(4.7) \quad \lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) = \lim_{x \downarrow l_1} \frac{P_x^*(\sigma_1 = \infty)}{s(x) - s(l_1)}
$$

$$
= \begin{cases} 
\frac{2}{a_2/[a_2(s(l_2) - s(l_1))] + b_2] } & \text{if } l_2 \text{ is } (s, m, 0)-\text{regular and the boundary condition (2.1) is posed at } l_2, \\
\frac{1}{\{s(l_2) - s(l_1)\}} & \text{otherwise.}
\end{cases}
$$

(4.6) is obtained by the same way as (7) in Section 6.2 of [4]. We show (4.7). By means of (4.4),

$$
\lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) = - \lim_{\xi \to \infty} \tilde{n}(\xi) \geq \lim_{x \downarrow l_1} \frac{P_x^*(\sigma_1 = \infty)}{s(x) - s(l_1)}.
$$

On the other hand, by (2.6) and (4.5),

$$
\lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) = \lim_{\xi \to \infty} \tilde{n}(\xi) \int_{0}^{\xi} d_\eta(e^{-\lambda \eta} - 1) \leq \lim_{\xi \to \infty} \int_{0}^{\xi} \tilde{n}(\eta) d_\eta(e^{-\lambda \eta} - 1)
$$

$$
= - \frac{D_s g_2^*(l_1, \lambda)}{g_2^*(l_1, \lambda)} + \frac{D_s g_2^*(\eta, \lambda)}{g_2^*(\eta, \lambda)} + \lambda \int_{(1, \infty)} g_2^*(\xi, \lambda) dm(\xi)
$$

$$
\leq \frac{1}{g_2^*(l_1, \lambda)} \frac{g_2^*(l_1, \lambda) - g_2^*(\eta, \lambda)}{s(\eta) - s(l_1)} + \lambda \{m(\eta) - m(l_1)\},
$$

for any $\eta > l_1$. Since the limit $\lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1)$ is independent of $\lambda$, letting $\lambda \downarrow 0$ and $\eta \downarrow l_1$, we get

$$
\lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) \leq \lim_{\eta \downarrow l_1} \frac{P_\eta^*(\sigma_1 = \infty)}{s(\eta) - s(l_1)}.
$$

Thus we obtain

$$
(4.8) \quad \lim_{\xi \to \infty} \tilde{n}(\xi)(e^{-\lambda \xi} - 1) = \lim_{x \downarrow l_1} \frac{P_x^*(\sigma_1 = \infty)}{s(x) - s(l_1)}.
$$

If $s(l_2) < \infty$ and $l_2$ is not $(s, m, 0)$-regular, or $l_2$ is $(s, m, 0)$-regular and absorbing, then

$$
P_x^*(\sigma_1 = \infty) = \frac{s(x) - s(l_1)}{s(l_2) - s(l_1)},
$$

and hence we have (4.7). Assume that $l_2$ is $(s, m, 0)$-regular and elastic. Since $g_2^*(x, 0)$ satisfies (2.5) and (2.6) with $\alpha = 0$,

$$
P_x^*(\sigma_1 < \infty) = \frac{g_2^*(x, 0)}{g_2^*(l_1, 0)} = \frac{g_2^*(x, 0) / g_2^*(l_2, 0)}{g_2^*(l_1, 0) / g_2^*(l_2, 0)} = \frac{1 + (a_2/b_2)(s(l_2) - s(x))}{1 + (a_2/b_2)(s(l_2) - s(l_1))},
$$

---132---
from which
\[
P^*_x(\sigma_{l_1} = \infty) = \frac{a_2\{s(x) - s(l_1)\}}{a_2\{s(l_2) - s(l_1)\} + b_2}.
\]
Combining this with (4.8), we obtain (4.7).

Since (4.7) coincides with \(\gamma^*\) given by (4.2), by virtue of (4.5), (4.6), and (4.7), we obtain
\[
-D_s g_2(l_1, \lambda) = \gamma^* + \int_0^\infty (1 - e^{-\lambda \xi}) d\tilde{n}(\xi).
\]

It is proved in Section 6.2 of [4] that the measure \(d\tilde{n}(\xi)\) has the density \(n^*(\xi)\) with respect to the Lebesgue measure \(d\xi\), and \(n^*(\xi)\) is given by (4.3). Thus the proof is complete.

**Remark 4.2** We note two representations of \(\gamma^*\). The first one is \(\gamma^* = \lim_{a \rightarrow 0} G^*(\alpha, l_1, l_1)^{-1}\), which is obtained by R. M. Blumenthal and R. K. Getoor (see Theorem 3.21 in Chapter V of [1]). It is easy to see that \(\lim_{a \rightarrow 0} G^*(\alpha, l_1, l_1)^{-1}\) coincides with the right hand side of (4.2).

The second one is \(\gamma^* = \mathcal{E}(\varphi, \varphi)\), which is given by M. Fukushima and H. Tanaka (see (2.21) of [3]), where \((\mathcal{E}, \mathcal{F})\) is the Dirichlet form corresponding to \(D^*_{s,m,0}\), \(l_2\) is absorbing or reflecting whenever it is \((s, m, 0)\)-regular, and \(\varphi(x) = P^*_x(\sigma_{l_1} < \infty)\). It is easy to see that \(\mathcal{E}(\varphi, \varphi)\) coincides with the right hand side of (4.2).

## 5 Inverse local time for \(D^*_{(h)}\)

When \(l_1\) is \((s_h, m_h, 0)\)-regular for \(D_{(h)}\), it is possible to pose the reflecting boundary condition at \(l_1\). Let \(D^*_{(h)} = [X(t), P^*_{(h)}]_{x,t}\) be such diffusion process on \(I\). We consider the local time \(l^*_{(h)}(t, \xi)\), that is, \(\int_0^t f(X(u)) du = \int_1 l^*_{(h)}(t, \xi) f(\xi) dm_h(\xi), \quad t > 0\), for bounded continuous functions \(f\) on \(I\). We denote by \(\tau^*_{(h)}(t)\) the inverse local time \(l^*_{(h)}^{-1}(t, l_1)\) at the end point \(l_1\).

The following result is an immediate consequence of Propositions 3.1 and 4.1.

**Proposition 5.1** Assume that \(l_1\) is \((s_h, m_h, 0)\)-regular and reflecting. Then there are a real number \(\gamma^*_{(h)}\) and Lévy measure density \(n^*_{(h)}(\xi)\) such that

\[
E^*_{(h),l_1} e^{-\lambda \tau^*_{(h)}(t)} = \exp\left\{-\gamma^*_{(h)} t - t \int_0^\infty (1 - e^{-\lambda \xi}) n^*_{(h)}(\xi) d\xi\right\},
\]

\[
\gamma^*_{(h)} = \begin{cases} 
\frac{a_{(h),2}}{[a_{(h),2}(s_h(l_2) - s_h(l_1))] + b_2} & \text{if } l_2 \text{ is } (s, m, 0)\text{-regular and the boundary condition (2.1) is posed at } l_2, \\
1/\{s_h(l_2) - s_h(l_1)\} & \text{otherwise,}
\end{cases}
\]

\[
n^*_{(h)}(\xi) = \lim_{x,y \rightarrow l_1} D_{s_h(x),D_{s_h(y)}l^*_{(h)}(\xi, x, y)}.
\]

Now we give a representation of \(n^*_{(h)}(\xi)\) by means of items corresponding to the diffusion process \(D_{s,m,k}\). By virtue of Section 3, \(l_1\) is \((s_h, m_h, 0)\)-regular if and only if one of the following conditions is satisfied.

\[
l_1 \text{ is } (s, m, k)\text{-regular and } h(l_1) \in (0, \infty).
\]

\[
l_1 \text{ is } (s, m, k)\text{-entrance, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty.
\]
Remark 5.3

\[
l_1 \text{ is } (s, m, k)\text{-natural, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty.
\]

In the same way as in Theorem 3.2 of [9], we obtain the following theorem. So we omit the proof.

**Theorem 5.2** Let \( h \in \mathcal{H}_{s,m,k,\beta} \). Assume one of (5.4), (5.5), and (5.6). Further assume that \( l_1 \) is reflecting. Then (5.1) holds true, and \( \gamma_n(l_1) \) and \( n_{l_1}(\xi) \) are given by (5.2) and (5.3), respectively. In particular, if (5.4) is satisfied, then

\[
n_{l_1}(\xi) = h(l_1)^2 e^{-\beta \xi} \int_{[0,\infty)} e^{-\xi \lambda} d\sigma^0(\lambda)
\]

\[
= h(l_1)^2 e^{-\beta \xi} \lim_{x,y \to l_1} D_{s(x)} D_{s(y)} p^0(\xi, x, y) = h(l_1)^2 e^{-\beta \xi} n^*(\xi),
\]

where \( n^*(\xi) \) is given by (4.3). If (5.5) is satisfied, then

\[
n_{l_1}(\xi) = D_s h(l_1)^2 e^{-\beta \xi} \int_{[0,\infty)} e^{-\xi \lambda} d\sigma^*(\lambda) = D_s h(l_1)^2 e^{-\beta \xi} \lim_{x,y \to l_1} p(\xi, x, y).
\]

**Remark 5.3** Let \( h \in \mathcal{H}_{s,m,0,\beta} \). Assume that \( l_1 \) is \((s, m, 0)\text{-regular and } h(l_1) \in (0,\infty)\). Then \( l_1 \) is also \((s_h, m_h, 0)\text{-regular. There exists a density function } q^*(\xi, x) = P^*_x(\sigma_{l_1} \in d\xi)/d\xi, \text{ which is given by}

\[
q^*(\xi, x) = \lim_{z \to l_1} \frac{p^0(\xi, z, x)}{s(z) - s(l_1)}.
\]

We put \( \mu_{\xi}(dx) = q^*(\xi, x) dm(x) \), which is the entrance law corresponding to the inverse local time \( \tau^*(t) \). Therefore the entrance law \( \mu(h, \xi)(dx) \) corresponding to \( \tau^*(h)(t) \) is given by

\[
\mu(h, \xi)(dx) = e^{-\beta \xi} h(l_1) h(x) \mu_{\xi}(dx).
\]

Assume that \( l_2 \) is absorbing or reflecting whenever it is \((s, m, 0)\text{-regular. Let } W' \text{ and } W \text{ be the spaces of excursions defined by}

\[
W' = \{ w : \exists \zeta(w) \in (0,\infty), \text{ } w \text{ is a continuous function from } (0, \zeta(w)) \text{ to } I \},
\]

\[
W = \{ w \in W' : \zeta(w) < \infty, \text{ then } \exists w(\zeta(w)-) \in \{ l_1, l_2 \} \}.
\]

Let \( n \) be the excursion law associated with the entrance law \( \{ \mu_{\xi} \} \). \( n \) is specified as follows.

\[
\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n)) n(dw) = \mu_{l_1} \left[ f_1 p_{l_2-l_1}^0 \left\{ f_2 \cdots p_{l_{n-1}-l_{n-2}}^0 \left( f_{n-1} p_{l_{n-1}-l_{n-2}}^0 \right) f_n \right\} \right]
\]

for \( 0 < t_1 < t_2 < \cdots < t_n \) and bounded measurable functions \( f_1, \cdots, f_n \) (see (4.4) of [3]). Therefore the excursion law \( n_{l_1}(w) \) associated with the entrance law \( \{ \mu(h, \xi) \} \) satisfies following.

\[
\int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_n(w(t_n)) n_{l_1}(dw) = e^{-\beta n} h(l_1) \int_W f_1(w(t_1)) f_2(w(t_2)) \cdots f_{n-1}(w(t_{n-1}))(f_n \circ h)(w(t_n)) n(dw).
\]
6 Examples

In this section, we present some interesting diffusion operators. Then, by means of Proposition 4.1 and Theorem 5.2, we deduce the corresponding Lévy measure densities. In the following, let \( l_1 = 0, l_2 = l < \infty \) and assume that \( s(x) = x \) and \( 0 < \rho < 1 \). \( C(\rho) \) is a positive number given by \( C(\rho) = \{ \rho(1 + \rho) \}^{\rho}/\Gamma(\rho) \).

Example 6.1 Let \( m(x) = C(l-x)^{-(1+1/\rho)} \), where \( C \) is a positive number. Namely,

\[
G = \hat{C}(l-x)^{2+1/\rho} \frac{d^2}{dx^2} \quad \text{on } (0,l),
\]

for a positive constant \( \hat{C} \). \( l_1 \) is \( (s,m,0) \)-regular and \( l_2 \) is \( (s,m,0) \)-natural. When we pose the reflecting boundary condition at \( l_1 \), we obtain the following by Proposition 4.1.

(6.1) \[
E_0^* \left[ e^{-\lambda \tau(t)} \right] = \exp \left\{ -t/l - t \int_0^\infty (1 - e^{-\lambda \xi}) n^*(\xi) \, d\xi \right\},
\]

(6.2) \[
n^*(\xi) = \int_0^\infty e^{-\lambda \xi} \sigma_0^0(\lambda) \, d\lambda,
\]

(6.3) \[
\sigma_0^0(\lambda) = \left[ l \rho \pi^2 \left\{ J_\rho \left( \sqrt{\lambda} z(l) \right)^2 + N_\rho \left( \sqrt{\lambda} z(l) \right)^2 \right\} \right]^{-1},
\]

where \( z(x) = 2\{C\rho(1+\rho)\}^{1/2}x^{-1/2\rho} \), \( J_\rho(z) \) are \( N_\rho(z) \) are Bessel functions, that is,

\[
J_\rho(z) = \left( \frac{z}{2} \right)^\rho \sum_{n=0}^\infty \frac{(-1)^n(z/2)^{2n}}{n! \Gamma(\rho + n + 1)}, \quad N_\rho(z) = \frac{1}{\sin(\rho \pi)} \left( J_\rho(z) \cos(\rho \pi) - J_{-\rho}(z) \right).
\]

We explain how to obtain \( \sigma_0^0(\lambda) \). \( \sigma_0^0(\lambda) \, d\lambda \) is the spectral measure in (2.7), which is given by the same argument as in [5], [6], [7], [13], [14].

Let \( \psi^0(x,\alpha) \) and \( \psi^*(x,\alpha) \) be the solutions of (2.10) and (2.11), respectively. For \( \alpha > 0 \) and \( 0 \leq x < l \), they are given as follows.

\[
\psi^0(x,\alpha) = 2\rho \sqrt{l(l-x)} \left\{ K_\rho \left( \sqrt{\alpha} z(l) \right) I_\rho \left( \sqrt{\alpha} z(l-x) \right) - I_\rho \left( \sqrt{\alpha} z(l-x) \right) K_\rho \left( \sqrt{\alpha} z(l-x) \right) \right\},
\]

\[
\psi^*(x,\alpha) = \left\{ \sqrt{(l-x)/l} \sqrt{\alpha} z(l) \left\{ K_{\rho+1} \left( \sqrt{\alpha} z(l) \right) I_\rho \left( \sqrt{\alpha} z(l-x) \right) + I_{\rho+1} \left( \sqrt{\alpha} z(l-x) \right) K_\rho \left( \sqrt{\alpha} z(l-x) \right) \right\},
\]

where \( I_\rho(z) \) and \( K_\rho(z) \) are the modified Bessel functions, that is,

\[
I_\rho(z) = \sum_{k=0}^\infty \frac{(z/2)^{\rho+2k}}{k! \Gamma(\rho + k + 1)}, \quad K_\rho(z) = \frac{\pi}{2} \frac{I_{-\rho}(z) - I_\rho(z)}{\sin \rho \pi}.
\]

Further we put

\[
\kappa(\alpha) = \lim_{x \to 0} \frac{\psi^0(x,\alpha)}{\psi^*(x,\alpha)} = \frac{2l \rho K_\rho(\sqrt{\alpha} z(l))}{\sqrt{\alpha} z(l) K_\rho(\sqrt{\alpha} z(l))}.
\]

The function \( \kappa(\alpha) \) is analytically continued to \( \mathbb{C} \setminus (-\infty,0] \), and the spectral measure \( \sigma^0 \) is defined by

\[
\sigma^0([\lambda_1,\lambda_2]) = -\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im \frac{1}{\kappa(\lambda - \sqrt{-1} \epsilon)} \, d\lambda,
\]

\[
-135-
\]
for all continuity points $\lambda_1$ and $\lambda_2$ of $\sigma^0(\lambda_1 < \lambda_2)$. Thus we find that $d\sigma^0(\lambda)$ has the
desity $\sigma^0_*(\lambda)$ with respect to the Lebesgue measure $d\lambda$, which is given by (6.3). Further
note the following

\begin{equation}
\psi^0(x, -\lambda) = \rho \pi \sqrt{l(l - x)} \left\{ - N_\rho \left( \sqrt{\lambda} z(l) \right) J_\rho \left( \sqrt{\lambda} z(l - x) \right) + J_\rho \left( \sqrt{\lambda} z(l) \right) N_\rho \left( \sqrt{\lambda} z(l - x) \right) \right\},
\end{equation}

for $0 \leq x < l$ and $\lambda > 0$. Then $p^0(t, x, y)$ is represented by

\[ p^0(t, x, y) = \int_{(0, \infty)} e^{-\lambda t} \psi^0(x, -\lambda) \psi^0(y, -\lambda) \sigma^0_*(\lambda) \, d\lambda. \]

Noting the asymptotic behavior of Bessel functions, we have

\[ \sigma^0_*(\lambda) \sim \frac{\lambda^{\rho/2} C(\rho)}{1 + \rho} \lambda^\rho \quad \text{as} \quad \lambda \to 0, \]

where $f(t) \sim g(t)$ as $t \to 0$ [resp. $t \to \infty$] stands for $\lim_{t \to 0 \text{[resp.}\, t \to \infty]} f(t)/g(t) = 1$ for
positive functions $f(t)$ and $g(t)$. Therefore we find

\[ n^*(\xi) \sim l^{-2} C'(\rho) \xi^{-(1 + \rho)} \quad \text{as} \quad \xi \to \infty. \]

For $\beta \geq 0$, let $h \in \mathcal{H}_{s, m, 0, \beta}$ and assume $0 < h(0) < \infty$. We denote by $\mathcal{G}_h$ by the
harmonic transform of $\mathcal{G}$ based on $h$, that is,

\[ \mathcal{G}_h = \mathcal{G} + 2\tilde{C}(l - x)^{2 + 1/\rho} \frac{h'(x)}{h(x)} \, dx = \tilde{C}(l - x)^{2 + 1/\rho} \left\{ \frac{d^2}{dx^2} + \frac{h'(x)}{h(x)} \frac{d}{dx} \right\} \quad \text{on} \quad (0, l). \]

Assume that 0 is reflecting. By means of Theorem 5.2, then (5.1) holds true, $\gamma_{(h)}^* = 1/\{s_h(l) - s_h(0)\}$, and $n_{(h)}^*(\xi)$ is given by

\[ n_{(h)}^*(\xi) = h(0)^2 e^{-\beta \xi} n^*(\xi), \]

where $n^*(\xi)$ is given by (6.2).

**Example 6.2** Let $m(x)$ satisfy $|m(0)| < \infty$ and

\[ m(l - 1/x) \sim x^{1 + 1/\rho} L(x) \quad \text{as} \quad x \to \infty, \]

where $L(x)$ is a slowly varying function. Since $l_1 = 0$ is $(s, m, 0)$-regular, we can define
the inverse local time $\tau^*(t)$ at 0 by putting the reflecting boundary condition. Let $K(x)$
be another slowly varying function such that

\[ \lim_{x \to \infty} K(x)^{1/\rho} L(x^{\rho} K(x)) = \lim_{x \to \infty} L(x^{\rho} K(x^{1/\rho} L(x))) = 1. \]

Then the Laplace transform of the distribution of $[\tau^*(t), \ t \geq 0]$ is given by the same
formula as (6.1). We discussed the asymptotic behavior of Lévy measure density cor-
responding to inverse local time in [10]. By means of Theorem 4 of [10], we find that the
Lévy measure density satisfies

\[ n^*(\xi) \sim l^{-2} C(\rho) \xi^{-(1 + \rho)} K(\xi)^{-1} \quad \text{as} \quad \xi \to \infty. \]

In this example we can consider the Lévy measure density corresponding to inverse local
time for the harmonic transformed process. The asymptotic behavior is given in The-
orem 8 of [10].
Acknowledgements

This research was partially supported by Grant-in-Aid for Young Scientists Young Scientists (B) No. 26800060 and by Grant-in-Aids for Scientific Research (C) No. 25400139.

References


Exponent of inverse local time for harmonic transformed process

TAKEMURA Tomoko and TOMISAKI Matsuyo

Abstract

We are concerned with inverse local time at regular end points for harmonic transform of a one dimensional diffusion process, and consider the corresponding exponents as well as the entrance law and the excursion law associated with inverse local time. In 1964 K. Itô and H. P. McKean showed that the Lévy measure density corresponding to the inverse local time at the regular end point for a recurrent one dimensional diffusion process is represented as the Laplace transform of the spectral measure corresponding to the diffusion process, where the absorbing boundary condition is posed at the end point. We demonstrate that their representation theorem is available for a transient one dimensional diffusion process, and deduce a representation theorem of the Lévy measure density corresponding to the inverse local time for a transient harmonic transformed process. Furthermore, we show a relation between exponents of inverse local time by means of 0-Green functions and those by means of Dirichlet forms, along with correlations between entrance laws of the original diffusion processes and its harmonic transform or between excursion laws and the harmonic transform. Moreover we present a new consideration for harmonic transform of non-minimal processes.