NON-HYPERBOLIC AUTOMATIC GROUPS AND GROUPS
ACTING ON CAT(0) CUBE COMPLEXES

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ABSTRACT. We discuss a problem posed by Gersten: Is every automatic group
which does not contain $\mathbb{Z} \times \mathbb{Z}$ subgroup, hyperbolic? To study this question,
we define the notion of “$n$-track of length $n$”, which is a structure like $\mathbb{Z} \times \mathbb{Z}$,
and prove its existence in the non-hyperbolic automatic groups with mild
conditions. As an application, we show that if a group acts freely, cellularly, properly
discontinuously and cocompactly on a CAT(0) cube complex and its
quotient is “weakly special”, then the above question is answered affirmatively.

1. Introduction

If a group $G$ has a finite $K(G,1)$ and does not contain any Baumslag-Solitar
groups, is $G$ hyperbolic? (See [3].) This is one of the most famous questions on
hyperbolic groups. Probably, many people expect that the answer is negative, and
it would be better to restrict our attention to some good class of groups. In this
paper we consider automatic groups. If an automatic group $G$ does not contain
any $\mathbb{Z} \times \mathbb{Z}$ subgroups, is $G$ hyperbolic? Our problem is listed in [2] and attributed
to Gersten.

Note that the class of all automatic groups contains the class of all hyperbolic
groups, all virtually abelian groups and all geometrically finite hyperbolic groups
[5]. A geometrically finite hyperbolic group is, in some sense, similar to hyperbolic
groups, but it might contain a $\mathbb{Z} \times \mathbb{Z}$ subgroup. Thus the class of automatic groups
is a nice target to consider the original question mentioned before.

Let us recall some related works very briefly. If the group is the fundamental
group of a closed 3-manifold, our question corresponds to the so-called “weak hy-
perbolization” of 3-manifolds [11]. Also, Wise proved that if the group satisfies the
small cancellation condition $B(6)$, then the above question is answered affirmatively
[18]. Papasoglu proved that the Cayley graph of a group which is semihyperbolic
but not hyperbolic contains a subset quasi-isometric to $\mathbb{Z} \times \mathbb{Z}$ [14].

In this paper, we define the notion of “$n$-track of length $n$”, which suggests a clue
of the existence of $\mathbb{Z} \times \mathbb{Z}$ subgroup, and show its existence in every non-hyperbolic
automatic groups with mild conditions.

As an application, we will show the next theorem:

**Theorem 5.4** Let $G$ be a group acting freely, cellularly, properly discontinuously
and cocompactly on a CAT(0) cube complex $X$. If each hyperplane in $G \setminus X$ embeds
and does not self-osculate, and $G$ is not word hyperbolic, then $G$ contains $\mathbb{Z} \times \mathbb{Z}$ as
a subgroup.

2000 Mathematics Subject Classification. Primary 20F65; Secondary 20F67, 57M07.
Key words and phrases. automatic group; hyperbolic group; CAT(0) cube complex.
This work was supported by JSPS KAKENHI Grant Number 23540088.
We remark that the assumption “no self-osculating hyperplanes” can be made weaker. See section 5 for the precise conditions we need. See also Sageev and Wise [16]. They considered a similar problem for groups acting on CAT(0) cube complexes, and introduced the notion of “facing triple”. We do not know the relation between our condition “no self-contact” in section 5 and theirs. We also note that Caprace and Haglund showed in [4] that a convex cocompact subgroup of a Coxeter group is hyperbolic if and only if it does not contain a $\mathbb{Z} \times \mathbb{Z}$ subgroup.

This paper is organized as follows. In section 2, we review definitions and some properties of hyperbolic groups and automatic groups. In section 3, we introduce the notion of “\(n\)-track” and show its existence. In section 4, we review automatic structure of groups which act freely,cellularly,properly discontinuously and co-compactly on CAT(0) cube complexes due to Niblo–Reeves [12], and prepare some lemmas. In section 5, we state and prove our main theorem.

2. Hyperbolic groups and automatic groups

In this section, we briefly review definitions and some properties of hyperbolic groups and automatic groups. We refer to [5] for the general theory.

Let \(G\) be a finitely generated group with a set of generators \(A\). In this paper, we will always assume that \(A\) is closed under the operation of taking inversed, i.e., \(A^{-1} = A\). The Cayley graph \(\Gamma := \Gamma(G,A)\) of \(G\) with respect to \(A\) is a directed, labeled graph defined as follows: the set of vertices is \(G\) itself. For \(g, h \in G\), there is a directed edge \((gh)\), source \(g\) and target \(h\), with label \(a \in A\) if and only if \(ga = h\).

Let \(w\) be a word over \(A\). A prefix of a word is any number of leading letters of that word. We denote by \(\ell(w)\) the word length of \(w\) and by \(w(t)\) the prefix of \(w\) with length \(t\). The image of \(w\) in \(G\) by the natural projection is denoted by \(w\).

In this paper, we denote by \(w(t_1, t_2)\) the subpath of the image of \(w\) by the natural projection on \(\Gamma\) from the vertex \(w(t_1)\) to the vertex \(w(t_2)\). The Cayley graph \(\Gamma\) is a metric space by its path metric. We denote this metric by \(d(g, h)\) for \(g, h \in G\).

**Definition 2.1.** A geodesic metric space is said to be hyperbolic (in the sense of Gromov [6]) if there is a number \(\delta > 0\) such that, for any triangle \(\triangle xyz\) with geodesic sides, the distance from a point on one side to the union of the other two sides is bounded by \(\delta\). A group \(G\) with a set of generators \(A\) is called word hyperbolic if the Cayley graph \(\Gamma(G, A)\) is hyperbolic.

It should be noted that the definition of word hyperbolic group does not depend on the choice of generators. One of the most important properties of word hyperbolic groups is the following theorem.

**Theorem 2.2.** If \(G\) contains a \(\mathbb{Z} \times \mathbb{Z}\) subgroup, then \(G\) can not be word hyperbolic. (See [1].)

Next, we recall the concept of automatic structure. Again, let \(G\) be a finitely generated group with a set of generators \(A\). We denote by \(\varepsilon\) the identity element of \(G\). A special letter \(\$ \not\in A\) is used to define the automatic structure of the group. A finite state automaton \(M\) over an alphabet \(A\) is a machine that determines “accept” or “reject” for a given word over \(A\). See [5] for details. The language given by all the accepted words of a finite state automaton \(M\) is denoted by \(L(M)\).

**Definition 2.3.** An automatic structure on \(G\) consists of a finite state automaton \(W\) over \(A\) and finite state automaton \(M_x\) over \((A \cup \{\$\}) \times (A \cup \{\}\)) for \(x \in A \cup \{\varepsilon\}\), satisfying the following conditions:
Definition 3.1.

For $w = x_1 x_2 \cdots x_n$ and $w' = x_1' x_2' \cdots x_m'$, where $x_i, x'_i \in A$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, as the string $(x_1, x_1') (x_2, x_2') \cdots$ defined over $(A \cup \{\$\}) \times (A \cup \{\$\})$. If the word length of $w$ is not equal to $w'$, we use the padding letter $\$$. The input for the automaton is $(x_1, x_1') (x_2, x_2') \cdots (x_n, x_n') (\$, x_{n+1}') \cdots (\$, x_m')$.

$W$ is called the word acceptor, and each $M_x$ is called a compare automaton for the automatic structure. An automatic group is one that admits an automatic structure.

Lemma 2.4 (Lemma 2.3.2 of [5]). If $G$ has an automatic structure, there is a constant $k$ with the following property: If $(w_1, w_2)$ is accepted by one of the automata $M_x$, for $x \in A \cup \{\varepsilon\}$, then $d(w_1(t), w_2(t)) < k$ for any integer $t \geq 0$.

Such a number $k$ is called a fellow traveler’s constant for the structure.

We define some properties of automatic structures which will be assumed later.

Definition 2.5. Let $W$ be the word acceptor of an automatic structure of a group $G$ with a set of generators $A$.

1. The automatic structure is prefix closed if, for every $w \in L(W)$, any prefix $w(t) (0 \leq t \leq \ell(w))$ is an element of $L(W)$.
2. The automatic structure has the uniqueness property if the natural projection from $L(W)$ to $G$ is injective, thus bijective.
3. The group is weakly geodesically automatic if any word $w \in L(W)$ is a geodesic with respect to the path metric of $G$.
4. The group is strongly geodesically automatic if $L(W)$ is equal to the set of all geodesic words.

In this paper, we investigate the relation between word hyperbolicity and automaticity for finitely generated groups. Here is a basic fact about the relation.

Theorem 2.6 (Theorem 2 in [13]). Any finitely presented group is word hyperbolic if and only if it is strongly geodesically automatic.

In the proof of the above theorem, the following lemma was proved.

Lemma 2.7 (Theorem 1.4 in [13]). If a group is weakly geodesically automatic and not hyperbolic, then for any sufficiently large $M > 0$, there exists a pair of geodesic words $(b_1, b_2)$ such that $\overline{b_1} = \overline{b_2}$ and $d\left(\overline{b_1(r)}, \overline{b_2(r)}\right) > M$ for some $r$.

See Figure 2. We call $(b_1, b_2)$ in Figure 2 $M$-thick bigon with side $b_1, b_2$.

3. Existence of an $n$-track in non-hyperbolic automatic groups

Let $G$ be an automatic group with automatic structure $(A, W, \{M_x\}_{x \in A \cup \{\varepsilon\}})$ where $A$ is a set of generators with $A^{-1} = A$, $W$ the word acceptor and $M_x$ the compare automaton for $x \in A \cup \{\varepsilon\}$. The following is the key concept in this paper.

Definition 3.1. Let $T = \{t_1, t_2, \ldots, t_n\}$ be a set of mutually disjoint $n$ paths of length $n$ in $\Gamma$. We call $T$ an $n$-track of length $n$ if there exist $2n$ words $w_1, w'_1, w_2, w'_2,$
\[ \ldots, w_n, w'_n \text{ of } L(W) \text{ and a positive integer } r \text{ such that } (w'_i, w_{i+1}) \text{ is accepted by some compare automaton for } i = 1, 2, \ldots, n-1, \text{ and that } t_i = w_i(r, r+n) = w'_i(r, r+n) \text{ for } i = 1, 2, \ldots, n. \text{ See Figure 1.} \]

**Figure 1.** A 4-track \( T = \{t_1, t_2, t_3, t_4\} \) and its related paths.

The purpose of this section is to prove the following theorem.

**Theorem 3.2.** Let \( G \) be a weakly geodesically automatic group whose automatic structure \((A,W,\{M_x\}_{x \in A \cup \{\varepsilon\}})\) is prefix closed and has the uniqueness property. If \( G \) is not hyperbolic, then it contains an \( n \)-track of length \( n \) for any \( n > 0 \).

**Proof.** Let \( k \) be a fellow traveler’s constant for the automatic structure and set \( M = 2k(n+1)^2 \). By Lemma 2.7, there exists an \( M \)-thick bigon in \( \Gamma \). We denote by \( b_1 \) and \( b_2 \) the two sides of this \( M \)-thick bigon, and by \( e \) and \( g \) the two end points. Without loss of generality, we may assume that \( e \) is the identity vertex \( \varepsilon \).

Since the automatic structure is weakly geodesically automatic, there exists a word \( p_0 \) in \( L(W) \) whose image \( p_0 \) in \( \Gamma \) is a geodesic from \( e \) to \( g \). Then, at least one of two bigons \( (p_0, b_1) \) and \( (p_0, b_2) \) is \((M/2)\)-thick. We denote this bigon by \( B = (p_0, b) \), where \( b = b_1 \) or \( b_2 \). (See Figure 2.)

**Figure 2.** An \( M \)-thick bigon.
By definition, we can find paths \( p_i \in L(W) \) from \( e \) to \( \tilde{b}(\ell(b) - i) \) for \( i = 0, 1, 2, \ldots, \ell(b). \) (See Figure 3.) Write \( P = \{p_j\}_{j=0}^{\ell(b)} \). Since the automatic structure is weakly geodesically automatic, each \( p_i \in P \) is geodesic and \( \ell(p_i) = \ell(b(\ell(b) - i)) = \ell(b) - i. \)

We claim that the intersection of two distinct paths \( p_i \) and \( p_j \) \( (i \neq j) \) of \( P \) is their common prefix (possibly the identity vertex \( e \)) only. To see this, suppose that \( \gamma_i \) and \( \gamma_j \) in \( \Gamma \) have an intersection \( p_i(t_i) = p_j(t_j) \) in \( G \). Since the automatic structure is prefix closed, both prefixes \( p_i(t_i) \) of \( p_i \) and \( p_j(t_j) \) of \( p_j \) are in \( L(W) \). Then, the uniqueness property implies that \( p_i(t_i) = p_j(t_j) \) (thus \( t_i = t_j \)) and the claim is proved.

Since \( B = (p_0, b) \) is an \((M/2)\)-thick bigon, there exists a number \( r \) such that \( d(p_0(r), b(r)) \geq M/2. \) Let \( \Lambda_j \) \( (j = 0, 1, 2, \ldots) \) be the graph whose vertex set \( V_j \) is the subset \( \{p_i(r + jn) \mid i = 0, 1, \ldots, \ell(b) - (r + jn) - 1\} \) of \( V(\Gamma) \), and whose edge set \( E_j \) is \( \{p_i(r + jn), p_{i+1}(r + jn) \mid i = 0, 1, \ldots, \ell(b) - (r + jn) - 1\} \). We consider \( \Lambda_j \) to be a shortest path in \( \Lambda_j \) from \( \tilde{b}(r + jn) \) to \( \tilde{b}(r) \). Let \( \gamma_j \) be a shortest path in \( \Lambda_j \) from \( \tilde{b}(r + jn) \). We set \( R = r + Jn. \)

Clearly, \( \Lambda_{j+1} \neq \emptyset \), and let \( \gamma_{j+1}^0, \gamma_{j+1}^1, \gamma_{j+1}^2, \ldots, \gamma_{j+1}^{L+1} \) be the geodesic paths in \( L(W) \) from \( e \) to the vertices of \( \gamma_{j+1} \). Note that \( \gamma_{j+1}^0 = p_0(R + n) \) and \( \gamma_{j+1}^{L+1} = p_{\ell(b) - (R + n)}(R + n) = p_{\ell(b)}. \) Let \( t^i = \gamma_{j+1}^i(R, R + n) \) \( (i = 0, \ldots, L) \). By construction, \( t^i \) and \( t^{i+1} \) are subpaths of \( p_m \) and \( p_{m+1} \) respectively for some \( m \), and \( (p_m, p_{m+1}) \) is accepted by some compare automaton. Let \( T' = \{t^0, t^1, \ldots, t^L\} \). If \( n \) consecutive paths in \( T' \) are mutually disjoint, then they give an \( n \)-track of length \( n \). Note that if some of \( T' \) intersect each other, this gives branches in the union of \( T' \) by the above claim. (See Figure 3.) Let \( y \) be the number of branches in the union of \( T' \), that is:

\[
y := \# \left\{ \gamma_{j+1}^i(R + n) \mid 0 \leq i \leq L \right\} - \# \left\{ \gamma_{j+1}^i(R) \mid 0 \leq i \leq L \right\},
\]

where \( \#X \) denotes the cardinality of a set \( X \).

![Figure 3. Finding an n-track of length n.](image)
We claim that \( y \leq n \). To see this, let \( \gamma_{J+1} \) be the image of \( \gamma_{J+1} \) by the natural projection \( \pi \) from \( \Lambda_{J+1} \) to \( \Lambda_J \), that is, \( \pi(\gamma_{J+1}(R+n)) = \gamma_{J+1}(R) \). For a path \( \gamma \) in a graph \( \Lambda \), let \( \ell_{\Lambda}(\gamma) \) denote the distance between the end points of \( \gamma \) in \( \Lambda \). Then, we have \( \ell_{\Lambda_J}(\gamma_{J+1}) \leq \ell_{\Lambda_{J+1}}(\gamma_{J+1}) - y \). Since there are \( n \) end-positions \( p_{\ell(b)-(R+n-1)}(R) = p_{\ell(b)-(R+n-1)}(R+1), \ldots, p_{\ell(b)-(R+n-1)}(R+n-1) \) of \( p_i \)'s between \( \Lambda_J \) and \( \Lambda_{J+1} \), and \( \gamma_J \) is a shortest path in \( \Lambda_J \), we have \( \ell_{\Lambda_J}(\gamma_J) \leq \ell_{\Lambda_J}(\gamma_{J+1}) + n \). (See Figure 4.) Since \( \gamma_J \) has the maximal length in \( \{ \gamma_0, \gamma_1, \gamma_2, \ldots \} \), we have \( \ell_{\Lambda_J}(\gamma_{J+1}) \leq \ell_{\Lambda_J}(\gamma_J) \). Therefore, \( \ell_{\Lambda_J}(\gamma_{J+1}) + y \leq \ell_{\Lambda_{J+1}}(\gamma_{J+1}) \leq \ell_{\Lambda_J}(\gamma_J) \leq \ell_{\Lambda_J}(\gamma_{J+1}) + n \), and it follows that \( y \leq n \) and the claim is proved.

![Figure 4. Branches and end points.](image)

Recall that we set \( M = 2k(n+1)^2 \) at the beginning of this proof, where \( k \) is a fellow traveler’s constant. Hence, we have \( \ell_{\Lambda_{J+1}}(\gamma_{J+1}) + n \geq \ell_{\Lambda_J}(\gamma_J) \geq \ell_{\Lambda_J}(\gamma_0) = M/2k = (n+1)^2 \). Recall that the number of elements of \( T' \) is \( L + 1 = \ell_{\Lambda_{J+1}}(\gamma_{J+1}) + 1 \). Since there are at most \( n \) branches in \( T' \) and the number of elements in \( T' \) is more than \( (n+1)^2 - n + 1 \) by the above inequality, there exist consecutive \( n \) paths in \( T' \) with the desired property, and the theorem is proved.

### 4. \textbf{CAT}(0) cube complexes

Does the existence of an \( n \)-track of length \( n \) for any \( n \) imply the existence of \( \mathbb{Z} \times \mathbb{Z} \) subgroup? We do not have the complete answer. But, as an application of the theorem in the previous section, we will give a partial answer to this question in the next section for the groups acting on \textbf{CAT}(0) cube complexes. In this section, we review Niblo and Reeves [12] and prepare some lemmas for the proof of the main theorem.

See [10] for \textbf{CAT}(0) and its relation to hyperbolicity. See also [9], [16].

#### 4.1. \textbf{The automatic structure for groups acting on CAT(0) cube complexes}

In this subsection, we briefly review the automatic structures given by Niblo and Reeves [12].

An \( n \)-cube is a copy of \([-1,1]^n\). A cube complex is obtained from a collection of cubes of various dimensions by identifying certain subcubes. A flag complex is a simplicial complex with the property that every finite set of pairwise adjacent vertices spans a simplex. Let \( X \) be a cube complex. The link of a vertex \( v \) in \( X \) is a complex built from simplices corresponding to the corners of cubes adjacent to \( v \).
Definition 4.1. A cube complex $X$ is nonpositively curved if, for each vertex $v$ in $X$, link$(v)$ is a flag complex.

Gromov [6] showed that a cube complex is CAT(0) if and only if it is simply connected and nonpositively curved. Many groups studied in combinatorial group theory act properly and cocompactly on CAT(0) cube complexes.

Let us recall the definition of hyperplane for cube complex. Our reference here is [7]. See also [8]. A midplane in a cube $[-1,1]^n$ is the subspace obtained by restricting exactly one coordinate to 0. For an edge in a cube, there is a unique midplane which cuts the edge transversely. A hyperplane $H$ of a cube complex $X$ is obtained by developing the midplanes in $X$, i.e., identifying common subcubes of midplanes which cuts the same edge. These edges are said to be dual to $H$.

This is a basic fact about hyperplane.

Lemma 4.2 (Proposition 2.7 in [12]). Every hyperplane in CAT(0) cube complex $X$ separates $X$ into exactly two components.

Each component is referred to as the halfspace associated with $H$.

Let $X$ be a CAT(0) cube complex and consider a sequence of cubes $\{C_i\}_{i=0}^n$ in $X$, each of dimension at least 1, such that each cube meets its successor in a single vertex $\tilde{v}_i = C_{i-1} \cap C_i$. This sequence is called a cube-path if $C_i$ is the cube of minimal dimension containing $\tilde{v}_i$ and $\tilde{v}_{i+1}$. Let $\tilde{v}_0$ be the vertex of $C_0$, which is diagonally opposite to $\tilde{v}_1$, and $\tilde{v}_{n+1}$ the vertex of $C_n$, diagonally opposite to $\tilde{v}_n$. $\tilde{v}_0$ is called the initial vertex and $\tilde{v}_{n+1}$ the terminal vertex. For a cube $C \in X$, $St(C)$ is the union of all cubes which contain $C$ as a subface (including $C$ itself).

Definition 4.3 (Definition 3.1 in [12]). A cube-path is called a normal cube-path if $C_i \cap St(C_{i-1}) = \tilde{v}_i$.

Lemma 4.4. Given two vertices $\iota, \tau \in V(X)$, there is a unique normal cube-path from $\iota$ to $\tau$ (Proposition 3.3 in [12]). A normal cube-path achieves the minimum length among all cube-paths joining the endpoints. (See remark in section 3 in [12] and [15].)

Remark 4.5 (Remark at the end of section 3 in [12]). Given a vertex $\tilde{v}$ on a normal cube-path, which terminates at $\tau$, the cube following $\tilde{v}$ is spanned by the planes which meet $St(\tilde{v})$ and separate $v$ from $\tau$.

Let $X$ be a CAT(0) cube complex, and $V(X)$ its vertex set. Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on $X$. Let $G \backslash X$ denote the quotient of the complex $X$ by the action of $G$. The fundamental groupoid $\pi(G \backslash X)$ is the groupoid whose objects are the points of $G \backslash X$ and morphisms between points $v, v'$ are homotopy classes of paths in $G \backslash X$ beginning at $v$ and ending at $v'$. The multiplication in $\pi(G \backslash X)$ is induced by composition of paths.

A directed cube is a cube with two ordered diagonally opposite vertices (the head and the tail) specified. Let $A$ be the set of homotopy classes of the diagonal of all directed cubes in $G \backslash X$. The correspondence between $A$ and directed cubes in $G \backslash X$ is one to one. The directed cubes in $X$ can be labelled equivariantly by (the lifts of) $A$, so each cube-path in $X$ defines a word over $A$. Let $\mathcal{L}$ be the set of words over $A$ which correspond to normal cube-paths.

Lemma 4.6. Let $A$ and $\mathcal{L}$ be as above. Then we have:
(1) There exists an isometry between $\pi(G\setminus X)$ with the word metric given by $A$ and $V(X)$ with the metric given by normal cube-paths (Lemma 4.1 in [12]).

(2) $L$ is regular over $A$ (Proposition 5.1 in [12]).

(3) $L$ satisfies 1-fellow travel property (Proposition 5.2 in [12]).

In particular, $(A, L)$ induces an automatic structure for $\pi(G\setminus X)$. (See Theorem 5.3 in [12].) This structure is prefix closed, weakly geodesically automatic with the uniqueness property ((1) and Lemma 4.4).

The set of states of (non-deterministic) finite-state automaton for $L$ is a $G$ (Proposition 5.1 in [12]). Thus, there is a natural map from the set of states of the word acceptor of $\pi(G\setminus X)$ to $G\setminus X$ by taking the tail of directed cubes. For vertices $\tilde{v}, \tilde{u}$ in $X$, we denote by $d(\tilde{v}, \tilde{u})$ the distance given by normal cube-paths.

Let $v$ be a vertex in $G\setminus X$. The group $G$ is realized as a subgroupoid $\pi(G\setminus X, \{v\})$ whose object is $v$ only, and whose morphisms are all the morphisms of $\pi(G\setminus X)$ between $v$. It is easy to construct an automatic structure for the group $G = \pi(G\setminus X, \{v\})$ from the automatic structure for the groupoid $\pi(G\setminus X)$.

### 4.2. Standard automata.

Let $G$ be a group or groupoid with automatic structure $M = (A, W, \{M_x\}_{x \in A \cup \{e\}})$, and $k$ a fellow traveler’s constant for $M$. For later purpose, we construct an automaton $M$ from $M$. It is called standard automata in [5] when $G$ is a group. (See Definition 2.3.3 in [5].) Put

$$S' := \{(s, t, g) \mid s, t \in S_W, s \neq F_W, g \in G, t(g) \leq k\},$$

where $S_W$ is the state set of $W$ and $F_W$ is the failure state of $W$. The state set $S$ of $M$ is $S' \cup \{\text{failure state} F\}$. The initial state of $M$ is $(s_0, s_0, id)$, where $s_0$ is the initial state of $W$. The transition function $\mu$ of $M$ is:

$$\mu((s, t, g), (x, y)) = \begin{cases} (\mu_W(s, x), \mu_W(t, y), x^{-1}g \ y) & \text{if it is in } S', \\ F & \text{otherwise,} \end{cases}$$

where $\mu_W$ is the transition function of $W$. Note that $\mu$ can be extended in a unique way to a map $A^* \times A^* \to S$, where $A^*$ is the set of all words over $A$, and we also denote it by $\mu$.

### 4.3. Groups acting on CAT(0) cube complexes.

Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on a CAT(0) cube complex $X$. Let $\tilde{v}$ be a vertex and $H$ a hyperplane in $X$. Let $\mathcal{H}^-$ be the halfspace associated with $H$ that does not contain $\tilde{v}$. We define the distance between $\tilde{v}$ and $H$ as $d(\tilde{v}, 3) = d + \frac{1}{2}$, where $d = \min\{d(\tilde{v}, \tilde{v}') | \tilde{v}' \in \mathcal{H}^-\} - 1$. We denote by $N(H) \cong H \times [-1, 1]$ the cubical neighborhood of $H$.

**Lemma 4.7.** If $d(\tilde{v}, H) \geq \frac{3}{2}$, there exists a hyperplane $H'$ such that $d(\tilde{v}, H') = \frac{1}{2}$ and $H'$ separates $\tilde{v}$ and $H$.

We remark that $d(\tilde{v}, H') = \frac{1}{2}$ if and only if $St(\tilde{v}) \cap H' \neq \emptyset$.

**Proof.** We denote the halfspace associated with $H$ that contains $\tilde{v}$ by $\mathcal{H}^+$.

We prove the lemma by induction on $d(\tilde{v}, H)$. Suppose that $d(\tilde{v}, H) = \frac{3}{2}$. Let $\tilde{v}'$ be a vertex in $\mathcal{H}^+ \cap N(H)$ such that $d(\tilde{v}, \tilde{v}') = 1$. Let $C_1$ be the cube spanned by $\tilde{v}'$ and $\tilde{v}$. There exists an edge $e$ in $C_1$ adjacent to $\tilde{v}'$, not contained in $N(H)$ such that the hyperplane $H'$ defined by $e$ (i.e., dual to $e$) separates $\tilde{v}'$ and $\tilde{v}$. Then, $H'$
Next, suppose that \( d + \frac{1}{2} = d(\tilde{v}, H) > \frac{3}{2} \). Let \( \tilde{v}' \) be a vertex in \( H^+ \cap N(H) \) such that \( d(\tilde{v}, \tilde{v}') = d \). Let \( C_1, \ldots, C_d \) be the normal cube-path from \( \tilde{v}' \) to \( \tilde{v} \). Denote the vertex \( C_1 \cap C_2 \) by \( \tilde{v}_1 \). As in the base case, there exists a hyperplane \( H'' \) that separates \( \tilde{v}_1 \) and \( \tilde{v}' \) and \( H'' \) does not intersect \( H \). Since \( C_1, \ldots, C_d \) is a normal cube-path, \( H'' \) separates \( \tilde{v} \) and \( H \), and \( d(\tilde{v}, H'') < d(\tilde{v}, H) \). By induction, there exists a hyperplane \( H' \) such that \( d(\tilde{v}, H') = \frac{1}{2} \) and \( H' \) separates \( \tilde{v} \) and \( H'' \). Clearly, \( H' \) separates \( \tilde{v} \) and \( H \) and the lemma is proved.

The next lemma is our key technical lemma.

**Lemma 4.8.** Let \( C_0, \ldots, C_n \) be a normal cube-path, and \( \tilde{v}_0, \ldots, \tilde{v}_{n+1} \) the vertices of this cube-path, that is, \( \tilde{v}_i = C_{i-1} \cap C_i \) for \( i = 1, \ldots, n \) with \( \tilde{v}_0 \) the initial vertex and \( \tilde{v}_{n+1} \) the terminal vertex. Let \( H \) be a hyperplane separating \( \tilde{v}_0 \) and \( \tilde{v}_{n+1} \). If \( d(\tilde{v}_0, H) = d + \frac{1}{2} \), then \( H \) separates \( \tilde{v}_d \) and \( \tilde{v}_{d+1} \).

**Proof.** We prove the lemma by induction on \( d \).

The base case \( d = 0 \) is trivial as \( d(\tilde{v}_0, H) = \frac{1}{2} \) and \( H \) meets \( St(\tilde{v}) \). Then, \( H \) separates \( \tilde{v}_0 \) and \( \tilde{v}_1 \) by Remark 4.5.

Now, consider the case \( d > 0 \). Assume that \( d(\tilde{v}_0, H) = d + \frac{1}{2} \) and \( H \) does not separate \( \tilde{v}_d \) and \( \tilde{v}_{d+1} \). Since \( d(\tilde{v}_0, H) = d + \frac{1}{2} \), \( H \) can not separate \( \tilde{v}_0 \) and \( \tilde{v}_d \). Hence, \( H \) separates \( \tilde{v}_{d+1} \) and \( \tilde{v}_n \). By Remark 4.5, we see that \( H \cap St(\tilde{v}_d) = \emptyset \). By Lemma 4.7, there exists a hyperplane \( H' \) such that \( H' \cap St(\tilde{v}_d) \neq \emptyset \) and \( H' \) separates \( \tilde{v}_d \) and \( H \). Then, \( H' \) separates \( \tilde{v}_d \) and \( \tilde{v}_n \), and it follows that \( H' \) separates \( \tilde{v}_d \) and \( \tilde{v}_{d+1} \). (See Remark 4.5 again.) By the induction hypothesis, \( d(\tilde{v}_0, H') \geq d + \frac{1}{2} \), so we have \( d(\tilde{v}_0, H') = d + \frac{1}{2} \). It follows that \( d(\tilde{v}_0, H) = d(\tilde{v}_0, H') = d + \frac{1}{2} \) and this is a contradiction.

Let \( \mathcal{M} \) be the standard automaton for the automatic structure of the groupoid \( \pi(G \setminus X) \) given in 4.1. We use the same symbols as in the previous subsection. Let \( (s, t, g) \) be a state in \( \mathcal{M} \). Since \( L \) (the set of words corresponding to normal cube-paths) satisfies 1-fellow travel property, \( g \) is in \( A \) (the set of generators). Recall that \( A \) consists of directed cubes in \( G \setminus X \). We define the dimension the the state \( (s, t, g) \), denoted by \( \dim(s, t, g) \), as the dimension of \( g \) as a (directed) cube. We also define \( \dim(\text{failure state } F) = +\infty \).

**Lemma 4.9.** For any transition \( (s', t', g') = \mu((s, t, g), (x, y)) \) in \( \mathcal{M} \), with \( x \neq \emptyset \) and \( y \neq \emptyset \), we have \( \dim(s', t', g') \geq \dim(s, t, g) \).

**Proof.** Fix \( t \in V(X) \) as the base point of \( X \). Let \( \tau_0, \tau_1 \) be two points in \( X \) such that \( d(\tau_0, \tau_1) = 1 \) and \( d(t, \tau_0) = d(t, \tau_1) \). Denote the vertices associated to the normal cube-paths joining them by \( t = \tilde{v}_0, \ldots, \tilde{v}_n = \tau_0 \) and \( t = \tilde{u}_0, \ldots, \tilde{u}_n = \tau_1 \).

Let \( H \) be a hyperplane and put \( d = d(t, H) \). If \( H \) separates \( \tau_0 \) and \( \tau_1 \), then, by Remark 4.5 and Lemma 4.8, \( H \) does not separate \( \tilde{v}_i \) and \( \tilde{u}_i \) for \( i = 0, \ldots, d \), and separates \( \tilde{v}_d \) and \( \tilde{u}_d \) for \( i = d + 1, \ldots, n \). If \( H \) does not separate \( \tau_0 \) and \( \tau_1 \), but separate \( t \) from both \( \tau_0, \tau_1 \), then \( H \) does not separate \( \tilde{v}_i \) and \( \tilde{u}_i \) for all \( i = 0, \ldots, n \), since \( H \) separates \( \tilde{v}_d \) and \( \tilde{v}_{d+1} \) as well as \( \tilde{u}_d \) and \( \tilde{u}_{d+1} \).

It follows that the dimension of the cube spanned by \( \tilde{v}_i \) and \( \tilde{u}_i \) is equal to the number of hyperplanes separating \( \tau_0 \) and \( \tau_1 \) having distance to \( t \) less than \( i \). Since this dimension is equal to the corresponding state in \( \mathcal{M} \), the dimensions of the states are monotone increasing under the transitions in \( \mathcal{M} \).
For vertices $v, \tilde{v}$ in $X$, we denote the set of hyperplanes separating $v$ and $\tilde{v}$ by $S(v, \tilde{v})$. When $d(v, \tilde{v}) = 1$, we denote by $[v, \tilde{v}]$ the label of the directed cube from $v$ to $\tilde{v}$. Since our automatic structure has the uniqueness property, after fixing a basepoint of $X$, there is a natural map $P : X \to S_W$, where $S_W$ is the state set of the word acceptor for $\pi_1(G\setminus X)$. We say that vertices $v, \tilde{u}, v', u'$ in $X$ correspond to a transition $(s', t', g') = \mu((s, t, g), (x, y))$ in $M$ if $P(v) = s, P(\tilde{u}) = t, P(v') = s', P(u') = t', [v, u] = g, [v, v'] = x, [u, u'] = y$ and $[v', u'] = g'$.

**Lemma 4.10.** Suppose that the vertices $v, \tilde{u}, v', u'$ in $X$ correspond to a transition $(s', t', g') = \mu((s, t, g), (x, y))$ in $M$ with $\dim(s', t', g') = \dim(s, t, g)$. Then, $v, \tilde{u}, v', u'$ span a cube in $X$ such that $S(v, \tilde{v}) = S(u, \tilde{u})$ and $S(v', \tilde{v'}) = S(u', \tilde{u'})$.

**Proof.** By the argument in the proof of Lemma 4.9, it is clear that $S(v, \tilde{u}) = S(v', \tilde{u'})$, and $S(v, \tilde{v}) = S(u, \tilde{u'})$. Since $S(v, \tilde{u}) = S(v, \tilde{u}) \cup S(u, \tilde{u'})$, we have $H \cap St(v) \cap \emptyset$ for each $H \in S(v, \tilde{u})$. By Lemma 2.15 in [12], there exists a cube that this union spans, and the lemma is proved.

**Lemma 4.11.** Consider two transitions $(s', t', g') = \mu((s, t, g), (x, y))$ and $(s'', t'', g'') = \mu((s, t, g), (x', y'))$ in $M$, and suppose that $\dim(s, t, g) = \dim(s', t', g') = \dim(s'', t'', g'')$. Then, $x = x'$ implies $y = y'$. Similarly, $y = y'$ implies $x = x'$.

**Proof.** We will prove that $x = x'$ implies $y = y'$. Note that we have $s' = s''$ in this case.

Suppose that vertices $v, \tilde{u}, v', u'$ in $X$ correspond to a transition $(s', t', g') = \mu((s, t, g), (x, y))$, and vertices $\tilde{v}, \tilde{u}, \tilde{v}', \tilde{u}'$ in $X$ correspond to a transition $(s', t'', g') = \mu((s, t, g), (x', y'))$ in $M$. Let $C_1, C_2, C_3$, and $C_4$ be the cubes spanned by $\{v, \tilde{v}\}, \{\tilde{v}, \tilde{u}\}, \{\tilde{v}, \tilde{v}', \tilde{u}, \tilde{u}'\}$, and $\{\tilde{v}, \tilde{v}', \tilde{u}, \tilde{u}'\}$, respectively. Then, in $\text{link}(v)$, the simplices corresponding to $C_3$ and $C_4$ are spanned by the simplices corresponding to $C_1$ and $C_2$. Recall that $\text{link}(v)$ was a flag complex, because $X$ is nonpositively curved. Thus, we have $C_3 = C_4$. Hence, $u' = u''$ and we have $y = y'$. □

5. The main result

In this section, we state and prove our main theorem. We use the same symbols as in the previous section. To prove the main theorem, we need a stronger conclusion than Lemma 4.11. In order to state the next lemma, let us introduce some notation. (See [7] for more details.)

Let $\bar{a}, \bar{b}$ be oriented edges having a common initial (or terminal) vertex $v$. Oriented edges $\bar{a}$ and $\bar{b}$ are said to directly osculate at $v$ if they are not adjacent in $\text{link}(v)$. Let $\bar{c}, \bar{d}$ be oriented edges such that the terminal vertex $v$ of $\bar{c}$ is equal to the initial vertex of $\bar{d}$. Oriented edges $\bar{c}$ and $\bar{d}$ are said to indirectly osculate at $v$ if they are not adjacent in $\text{link}(v)$. Let $e, f$ be (unoriented) edges having a common end point $v$. Edges $e$ and $f$ are said to osculate at $v$ if they are not adjacent in $\text{link}(v)$.

We consider hyperplanes in $G\setminus X$. From now on, we assume that each hyperplane in $G\setminus X$ embeds.

A hyperplane $H$ is said to be 2-sided if its open cubical neighborhood is isomorphic to the product $H \times (-1, 1)$. If a hyperplane is not 2-sided, then it is said to be
1-sided. If $H$ is 2-sided, one can orient dual edges in a consistent way. A 2-sided hyperplane is said to directly self-osculate if it is dual to distinct oriented edges that directly-osculate. We say that 1-sided hyperplane self-osculates if it is dual to distinct (unoriented) edges that osculate.

In this paper, we introduce the following notion:

**Definition 5.1.** We say that a 2-sided hyperplane $H$ self-contacts if there are two vertices $u,v$ such that $d(u,v) = 1$ and $H$ directly self-osculates at $u$ and $v$. (See Figure 5.) We say that a 1-sided hyperplane $H$ self-contacts if there are two vertices $u,v$ such that $d(u,v) = 1$ and $H$ self-osculates at $u$ and $v$.

![Figure 5. A self-contacting hyperplane.](image)

Let $H$ be a hyperplane in $G \setminus X$. Let $M$ be a midplane of $H$, and $C = M \times [-1,1] \subset N(H)$ the cubical neighborhood of $M$. Let $u$ and $v$ be two vertices of $C$ not separated by $M$. We say that $C$ is spanned by $u$, $v$ and $H$ if $C$ is a cube of minimal dimension of this type, i.e., the cubical neighborhood of some midplane of $H$ that contains $u$ and $v$. If every hyperplane embeds and there is no hyperplane that self-contacts, then $C$ can be determined uniquely by $u$, $v$ and $H$ (and the choice of the side of $H$ if $H$ is 2-sided and “indirectly self-contacts”, i.e., having indirect osculations). (See Figure 5.)

**Remark 5.2.** By definition, if a cube complex is special in the sense of [7], then each hyperplane embeds, and it has no hyperplane of self-contact.

Let $P_{s,t,g}$ be the set of pairs of letters that may appear in a word of $L(M_{s,t,g})$, where $M_{s,t,g}$ is the automaton with the same set of states and transition as $M$ but having the initial state $(s,t,g)$, and accept states $\{ (s',t',g') | \dim(g') = \dim(g) \}$.

In other words, $(x,y)$ is in $P_{s,t,g}$ if there exists a sequence $(x_0,y_0), \ldots, (x_{n-1},y_{n-1})$, $(x,y)$ which is accepted by $M_{s,t,g}$. (Note that $P_{s,t,g}$ depends only on the strongly connected component of $M$.)

**Lemma 5.3.** If each hyperplane in $G \setminus X$ embeds and does not self-contact, then, for each $(s,t,g)$ with $g \neq \text{id}$, there exist two subsets $A', A'' \subset A$ and a bijection $f : A' \to A''$ such that

$$P_{s,t,g} = \{ (x,y) \in A' \times A'' | f(x) = y \}.$$ 

**Proof.** $A'$ and $A''$ are determined by taking first and second projections of $P_{s,t,g}$.

It suffices to show that if there are two sequences $(x_1,y_1), \ldots, (x_n,y_n)$ and $(x'_1,y'_1), \ldots, (x'_m,y'_m)$ both accepted by $M_{s,t,g}$, then $x_n = x'_m$ implies $y_n = y'_m$, as well as $y_n = y'_m$ implies $x_n = x'_m$. In this case, we can define $f(x_n) = y_n$. We prove the former case.
Set $s_0 = s'_0 = s$, $t_0 = t'_0 = t$, and $g_0 = g'_0 = g$. Define $(s_i, t_i, g_i)$ and $(s'_j, t'_j, g'_j)$ inductively by

$$(s_i, t_i, g_i) = \mu((s_{i-1}, t_{i-1}, g_{i-1}), (x_i, y_i)),$$
$$(s'_j, t'_j, g'_j) = \mu((s'_{j-1}, t'_{j-1}, g'_{j-1}), (x'_j, y'_j))$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let $v, u, v_x, v_y, v'_x, v'_y$ be vertices in $G \setminus X$ corresponding to $s, t, s_x, s_y$ and $t'_x$, $t'_y$ respectively. (For the correspondence between the set of states of the word acceptor and $G \setminus X$, see the paragraph after Lemma 4.6.) Put $d = \dim(s, t, g)$, and denote the hyperplanes separating $v$ and $u$ by $\{H_1, \ldots, H_d\} = S(v, u)$. (Recall that by Lemma 4.10, $S(v, u) = S(v_i, u_i)$ for all $i$ and $j$.) Let $C_k$ be the cube spanned by $v_{n-1}, v_n$ and $H_k$ for $k = 1, \ldots, d$. Recall that, by the assumption that each hyperplane embeds and does not self-contact, each $C_k$ is uniquely determined by $v_{n-1}, v_n$ and $H_k$ (and the obvious choice of the side of $H_k$ if needed). Let $C$ be the cube spanned by $v_{n-1}, v_n, u_{n-1}$, and $u_n$. Then, $C$ is spanned by $C_1, \ldots, C_d$ ($C$ contains $C_1, \ldots, C_d$), and it is uniquely determined by $v_{n-1}, v_n$ and $S(v, u)$. Define $C'$ for $v'_{m-1}, v'_m, u'_m$, and $u'_m$, in the same way. It is uniquely determined by $v'_{m-1}, v'_m$ and $S(v, u)$. If $x_n = x_m$, then we have $v_{n-1} = v'_{m-1}$, $v_n = v'_m$ and $C = C'$. Thus, $u_{n-1} = u'_{m-1}$ and $u_n = u'_m$. Hence we have $y_n = y'_m$. \( \square \)

This is our main theorem.

**Theorem 5.4.** Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on a CAT(0) cube complex $X$. If each hyperplane in $G \setminus X$ embeds and does not self-contact and $G$ is not word hyperbolic, then $G$ contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup.

**Proof.** First, note that Theorem 3.2 works for the groupoid $\pi(G \setminus X)$. By Lemma 4.6, itsNiblo–Reeves automatic structure described in the previous section is prefix closed, weakly geodesic and has the uniqueness property. $X$ is not hyperbolic since $X$ and the Cayley graph of $G$ are quasi-isometric. Hence, there exists an $n$-track of length $n$ in $X$ for any $n > 0$.

For two vertices $\tilde{v}, \tilde{v}'$ in $X$ with $d(\tilde{v}, \tilde{v}') = 1$, we denote by $\dim(\tilde{v}, \tilde{v}')$ the dimension of the cube spanned by $\tilde{v}$ and $\tilde{v}'$. (Recall that we use normal cube-paths to define our metric.)

Fix a vertex $v$ as the base point in $X$. Let $T = \{t_1, \ldots, t_n\}$ be an $n$-track of length $n$ in $X$. We will improve $T$ in two steps.

**Step one.** The vertices in $T$ can be identified with $\{1, 2, \ldots, n\} \times \{0, 1, \ldots, n\}$, where $(i, j)$ corresponds to the $j$-th vertex in $t_i$, which was denoted by $v_i(r + j)$ in section 3. Here, we denote this vertex by $v_{i,j}$. We claim there exists an $m$-subtrack of length $m$ of $T$ of the form $T' = \{i_0, i_0 + 1, \ldots, i_0 + m - 1\} \times \{j_0, j_0 + 1, \ldots, j_0 + m\}$ such that

$$\dim(v_{i,j}, v_{i+1,j}) = \dim(v_{i,j+1}, v_{i+1,j+1}) \text{ for any } (i, j) \in T'. \quad (1)$$

By Lemma 4.9, for each $i \in \{1, \ldots, n\}$, the number of indices $j \in \{0, \ldots, n\}$ with $\dim(v_{i,j}, v_{i+1,j}) \neq \dim(v_{i,j+1}, v_{i+1,j+1})$ is smaller than the maximal dimension of cubes in $G \setminus X$. Thus, it is easy to see that for any $m > 0$, there exists $n$ such that any $n$-track of length $n$ contains an $m$-track of length $m$ that satisfies (1). By abusing notation, we refer to this subtrack by the same symbol $T = \{t_1, \ldots, t_n\}$ and denote
its size by \( n \). (Note that we do not assume that \( \dim(v_{i,j}, v_{i+1,j}) = \dim(v_{i+1,j}, v_{i+2,j}) \) in \( T \).) Step one is finished.

**Step two.** Note that, even though each \( t_i \) is a normal cube-path by definition, the path given by \( v_{1,n}, v_{2,n}, \ldots, v_{n,n} \) (located at “far end” of the \( n \)-track constructed in Step one) may not be a normal cube-path. The aim of this step is to replace this path by a normal cube-path. This condition will be used in the last three paragraphs of this proof.

Recall that, for two vertices \( \tilde{v}, \tilde{v}' \) in \( X \), we denote by \( S(\tilde{v}, \tilde{v}') \) the set of hyperplanes separating them. By the condition (1), for each \( i \in \{1, \ldots, n-1\} \), there exists a set of hyperplanes \( \mathcal{F}_i \) such that \( S(v_{i,j}, v_{i+1,j}) = \mathcal{F}_i \) for any \( j \in \{0, \ldots, n\} \).

By Lemma 4.8, for each hyperplane \( H \in \bigcup \mathcal{F}_i \), we have \( d(i, H) < d(i, v_{1,0}) \). Now, put \( m = d(v_{1,n}, v_{n,n}) + 1 \) and let \( C_1, \ldots, C_{m-1} \) be the normal cube-path from \( v_{1,n} \) to \( v_{n,n} \). We denote the vertices of this normal cube-path by \( v'_1, v'_2, \ldots, v'_m \).

For \( i \in \{1, \ldots, m\} \), let \( t'_i \) be the postfix (tail) of the normal cube-path from the base point \( i \) to \( v'_i \) with length \( m \). See Figure 6.

We claim that \( T' = \{t'_1, \ldots, t'_m\} \) is an \( m \)-track of length \( m \).

First, we show that \( d(i, v'_i) = d(i, v_{1,1}) \) for \( i = 1, \ldots, m \). Put \( N = d(i, v_{1,1}) = d(i, v_{n,n}) \). Suppose that \( d(i, v'_i) > N \). Let \( H \) be a hyperplane in \( S(v, v'_i) \), where \( v \) is the vertex in \( t'_i \) with \( d(v, v'_i) = 1 \). Then we have \( d(i, H) = N + \frac{1}{2} \) and \( H \) separates \( v'_i \) from both \( v_{1,n} \) and \( v_{n,n} \). Then the normal cube-path \( C_1, \ldots, C_{m-1} \) intersects \( H \) more than one time which is a contradiction (Remark 4.5). Suppose that \( d(i, v'_i) < N \). Recall that, by Lemma 4.10, we have \( S(v_{1,n-1}, v_{1,1}) = S(v_{n,n-1}, v_{n,n}) \). Let \( H \) be an elements of this set. Then \( d(i, H) = N - \frac{1}{2} \), and \( H \) separates \( v'_i \) from both \( v_{1,n} \) and \( v_{n,n} \) which is a contradiction as in the previous case.
Next, let $\mathcal{H}_i'$ be the set of hyperplanes separating $v'_i$ and $v'_{i+1}$. Then,

$$
\bigcup_{i=1}^{m-1} \mathcal{H}_i' = S(v_{1,n}, v_{n,n}) \subset \bigcup_{i=1}^{n-1} \mathcal{H}_i.
$$

Thus, we have

$$
d(\iota, H') < d(\iota, v_{1,0})
$$

for each hyperplane $H' \in \cup_i \mathcal{H}_i'$. Thus, by Lemma 4.8, we have $t'_i \cap t'_j = \emptyset$ if $i \neq j$, and $T'$ is an $m$-track of length $m$. Moreover, (2) implies that $T'$ satisfies the condition (1).

The size $m$ can be smaller than $n$. But, since $X$ is locally finite, if $n$ is large enough, we may assume that $m = d(v_{1,n}, v_{n,n}) + 1$ is as large as we want. By abusing notation, we refer to this new track by the same symbol $T$ and denote its size by $n$, and vertices by $v_i$. In particular, $v'_1, \ldots, v'_m$ will now be called $v_{1,n}, \ldots, v_{n,n}$.

Step two is finished.

We denote by $t_{i,j}(\in A)$ the label of the directed edge from $v_{i,j}$ to $v_{i,j+1}$. Fix $i \in \{1, \ldots, n\}$, and let us consider a pair of consecutive tracks $t_i$ and $t_{i+1}$ in $T$. By definition, for each pair of vertices $(v_{i,j}, v_{i+1,j})$ on $t_i$ and $t_{i+1}$, there is a corresponding state $(s_j, t_j, g_j) \in \mathcal{M}$. (It is unique by the uniqueness property of the automatic structure.) Since $\dim(v_{i,j}, v_{i+1,j})$ is constant for all $j$, each state $(s_j, t_j, g_j)$ can be regarded as a state in $\mathcal{M}_{v_0, v_{n,0}}$, and clearly, $g_0 \neq id$. (For the definition of $\mathcal{M}_{v_0, v_{n,0}}$, see the paragraphs after Remark 5.2.) By Lemma 5.3, there exists a bijection $f_{i,i+1} : A \rightarrow A$ such that $t_{i+1,j} = f_i(t_{i,j})$ for any $j$. (In Lemma 5.3, the map was from a subset $A'$ of $A$ to another subset $A''$, but it is easy to extend this map from $A$ to $A$. The extension is not unique but the complement of $A'$ will not be used anyway.) By combining the $f_{i,i+1}$, for each $i, i' \in \{1, \ldots, n\}$, we have a bijection $f_{i,i'} : A \rightarrow A$ such that $t_{i',j} = f_{i,i'}(t_{i,j})$ for any $j$.

For vertices $\tilde{v}, \tilde{v}' \in X$, we denote the word over $A$ given by the normal cube-path from $\tilde{v}$ to $\tilde{v}'$ by $[\tilde{v}, \tilde{v}']$.

We claim that, if $n$ (the size of our $n$-track) is large enough, $T$ contains indices $I = \{i_0, i_1\}$ with $i_0 < i_1$ and $J = \{j_0, j_1\}$ with $j_0 < j_1$ that satisfy the following conditions:

(C1) $f_{i_0,i_1} : A \rightarrow A$ is the identity map.

(C2) $v_{i_0,j_0}$ and $v_{i_0,j_1}$ correspond to the same state in the word acceptor.

(C3) $v_{i,j}$ projects to the same vertex, say $v$, in $G\setminus X$ for all $i \in I, j \in J$.

(C4) The equality $[v_{i_0,j_0}, v_{i_1,j_0}] = [v_{i_0,j_1}, v_{i_1,j_1}]$ holds.

(C5) The two letters $[v_{i_0,j_0}, v_{i_0+1,j_0}]$ and $[v_{i_1,j_0}, v_{i_1+1,j_0}]$ coincide.

To see the claim, observe that

1. the number of permutations $A \rightarrow A$ is finite,
2. the number of states in the word acceptor is finite,
3. the number of vertices in $G\setminus X$ is finite,
4. the word length of $[v_{i_0,j}, v_{i_1,j}]$ is less than or equal to $|i_0 - i_1|$, and the bound is independent of $j$,
5. the number of elements of $A$ (the set of letters) is finite,

and these numbers do not depend on the choice of $T$. Then, it is easy to calculate the value of $n$ needed so that $T$ contains $I$ and $J$ which satisfy the above conditions. From now on, we assume that $n$ is large enough so that it satisfies these conditions.
Now, define $a = [v_{i_0,j_0}, v_{i_0,j_1}]$ and $b = [v_{i_0,j_0}, v_{i_1,j_0}] = [v_{i_0,j_1}, v_{i_1,j_1}]$. (For $b$, we used (C4).) We consider that these elements are in $\pi_1(G \setminus X, v)$ because of (C3). Note that these elements connect vertices $v_{i,j}$ ($i \in I, j \in J$) “vertically” and “horizontally.” By (C1) and (C4), we have $ab = ba$. (See Figure 7.) Since any isometry of a CAT(0) space of finite order must have a fixed point, both $a$ and $b$ have infinite order. To complete the proof, it suffices to show that $\langle a, b \rangle$ is rank two.

Let $H_0$ be an element of $S(v_{i_0,j_0}, v_{i_0+1,j_0}) = S(v_{i_0,n}, v_{i_0+1,n})$. (The equality holds because of step one.) We want to consider the action of elements in $\pi_1(G \setminus X)$ on hyperplanes. For the sake of simplicity, after conjugation, suppose that the vertex $v_{i_0,j_0}$ is the base point of $X$ as the Cayley graph of $\pi_1(G \setminus X)$. Denote by $H_1 = b(H_0)$ the image of $H_0$ by the action of $b$. By (C5), the image of the cube spanned by $v_{i_0,j_0}$ and $v_{i_0+1,j_0}$ by this action is the one spanned by $v_{i_1,j_0}$ and $v_{i_1+1,j_0}$. It follows that $H_1 \subset S(v_{i_1,j_0}, v_{i_1+1,j_0}) = S(v_{i_1,n}, v_{i_1+1,n})$. Then, we have $H_0 \neq H_1$, because $v_{1,n}, v_{2,n}, \ldots, v_{n,n}$ are the vertices of a normal cube-path (after step two) and every hyperplane separates a normal cube-path at most once (Remark 4.5). Let $\overline{H}$ be the image of $H_0$ (and therefore, of $H_1$) in $G \setminus X$. Since $\overline{H}$ is embedded, $H_0$ and $H_1$ do not intersect in $X$.

Let $H$ be a halfspace of $H_0$ containing $v_{n,n}$. By (C5), $b(H)$ also contains $v_{n,n}$. Since $H_0 \cap b(H_0) = \emptyset$, we have $b(H) \subset H$ (Lemma 4.2). Thus, the positive powers of $b$ gives a nested sequence of halfspaces $H \supset b(H) \supset b^2(H) \supset \cdots$. If $\langle a, b \rangle$ is cyclic, then there exist positive integers $p, q$ such that $a^p = b^q$. Since $b^q \in b^{q-1}(H)$, we have $a^p \in b^{q-1}(H) \subset H$. By (C2), $a^k$ (possibly with some prefix) is accepted by the word acceptor for any $k > 0$. Hence, $a^p$ is a normal cube-path. Observe that $St(v_{i_0,j_0}) \cap H_0 \neq \emptyset$, and the path $a = [v_{i_0,j_0}, v_{i_0,j_1}]$ is in the complement of $H$. This contradicts to $a^p \in H$ (Remark 4.5). Thus, this subgroup is not cyclic, but is of rank two.

Hence $\langle a, b \rangle$ is a $\mathbb{Z} \times \mathbb{Z}$ subgroup, and the theorem is proved. $\square$

Finally, we ask questions that we hope are interesting.
Question 5.5. Is the condition “no hyperplane of self-contact” necessary?

Question 5.6. A more general method is known to show that a group acting on a space is automatic. See [17]. Can one generalize the above result for this setting?

Acknowledgements

The authors would like to thank the referee who pointed out some errors in the original manuscript and gave helpful and useful suggestions.

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