Upper bounds for the Roman bondage number of graphs on closed surfaces

KATAGIRI Minyo∗

1 Introduction

Let $G$ be a simple graph, and its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. We denote the set of neighbors of a vertex $v$ of $G$ by $N_G(v)$. The degree $\deg(v)$ of a vertex $v$ denotes the number of neighbors of $v$ in $G$, $\Delta(G)$ is the maximum degree of $G$, and $\delta(G)$ is the minimum degree of $G$.

A set $D \subset V(G)$ is the dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$. Clearly, for any spanning subgraph $H$ of $G$, $\gamma(H) \geq \gamma(G)$. The bondage number of $G$, denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subset E(G)$ such that $\gamma(G - B) > \gamma(G)$, where $G - B$ is the graph with $V(G - B) = V(G)$ and $E(G - B) = E(G) \setminus B$. On bondage number of graphs, see ([5]).

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a Roman dominating function if every vertex $v$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) = 2$. The weight of a Roman dominating function is the value $\sum_{v \in V(G)} f(v)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function of $G$. The Roman bondage number $b_R(G)$ of a graph $G$ is the cardinality of a smallest set of edges $B \subset E(G)$ for which $\gamma_R(G - B) > \gamma_R(G)$, where $V(G - B) = V(G)$ and $E(G - B) = E(G) \setminus B$. On Roman bondage number of graphs, see ([1], [3], [4]).

In this paper, we give an upper bounds for the Roman bondage number of graphs on closed surfaces.

* Faculty (Natural Sciences, Mathematics), Assistant Professor
2 The Roman bondage number on closed surfaces

In this section, we consider an upper bounds for the Roman bondage number of graphs on closed surfaces.

N. J. Rad and L. Volkmann proved the following result ([4]).

\textbf{Theorem 2.1} If $G$ is a graph, $u, v, w \in V(G)$, and $uvw$ a path of length 2 in $G$, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - 3 - |N_G(u) \cap N_G(v)|$$

where $|X|$ is the cardinality of a set $X$.

\textbf{Corollary 2.2} Let $G$ be a graph, then

$$b_R(G) \leq \delta(G) + 2\Delta(G) - 3.$$

N. J. Rad and L. Volkmann also proved the following result ([3]).

\textbf{Theorem 2.3} If $G$ is a connected planar graph of order $\geq 3$, then $b_R(G) \leq 2\Delta(G)$.

We generalize this result for graphs on closed surfaces, \textit{cf} ([2]). A graph $G$ is 2-cell embeddable on a closed surface $M$ if it admits on the surface with no crossing edges, and $M \setminus G$ is homeomorphic to a union of disks. Such a drawing of $G$ on the closed surface $M$ is called an embedding of $G$ on $M$. The set of faces of an embedding of $G$ on $M$ is denoted by $F(G)$.

\textbf{Theorem 2.4} Let $G$ be a connected graph having maximum vertex degree $\Delta(G)$ and embeddable on a closed surface $M$ with Euler characteristic $\chi(M)$, then

$$b_R(G) \leq 2\Delta(G) + 1 - \left\lfloor \frac{\chi(M)}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor symbol.

\textit{Proof.}

Suppose $G$ is 2-cell embedded on a closed surface $M$. If $G$ has a vertex of degree $4 - \frac{1}{2}\chi(M)$ or less, we have $\delta(G) \leq 4 - \frac{1}{2}\chi(M)$,

$$b_R(G) \leq \delta(G) + 2\Delta(G) - 3 \leq 2\Delta(G) + 1 - \frac{\chi(M)}{2}.$$
and the above inequality holds. Therefore, we may assume that $\Delta(G) \geq \delta(G) \geq 5 - \frac{1}{2}\chi(M)$.

Suppose $b_R(G) \geq 2\Delta(G) + 2 - \frac{1}{2}\chi(M)$. Then, by Theorem 2.1, for any edge $e = uv \ (u, v \in V(G))$, we have

$$\deg(u) + \deg(v) + \Delta(G) - 3 - |N_G(u) \cap N_G(v)| \geq b_R(G)$$

$$\geq 2\Delta(G) + 2 - \frac{\chi(M)}{2}.$$  

This gives

$$\deg(u) + \deg(v) \geq \Delta(G) + 5 - \frac{\chi(M)}{2} + |N_G(u) \cap N_G(v)|.$$  

We may assume that $\deg(u) \leq \deg(v)$. If $\deg(u) = 5 - \frac{1}{2}\chi(M)$, then $\deg(v) \geq \Delta(G) + |N_G(u) \cap N_G(v)|$ so $\deg(v) = \Delta(G)$, and $|N_G(u) \cap N_G(v)| = 0$. Let $f$ and $f'$ be a face on side of $uw = e \in E(G)$, and $\text{gon}(f)$ the number of edges on the boundary of the face $f$. $u$ and $v$ cannot have any common neighbors, so $\text{gon}(f) \geq 4$ and $\text{gon}(f') \geq 4$. In this case

$$|E(G)| \geq \frac{\delta(G)|V(G)|}{2} \geq \frac{1}{2} \left( 5 - \frac{\chi(M)}{2} \right) \left( 6 - \frac{\chi(M)}{2} \right)$$

$$= \frac{(10 - \chi(M))(12 - \chi(M))}{8},$$

therefore

$$\frac{1}{\deg(u)} + \frac{1}{\deg(v)} + \frac{1}{\text{gon}(f)} + \frac{1}{\text{gon}(f')} - 1 - \frac{\chi(M)}{|E(G)|}$$

$$\leq + \frac{2}{10 - \chi(M)} + \frac{2}{10 - \chi(M)} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{8\chi(M)}{(10 - \chi(M))(12 - \chi(M))}$$

$$= \frac{-\chi(M)^2 - 2\chi(M) - 24}{2(10 - \chi(M))(12 - \chi(M))}$$

$$= \frac{-(\chi(M) + 1)^2 - 23}{2(10 - \chi(M))(12 - \chi(M))} < 0.$$ 

Suppose $\deg(u) = 6 - \frac{1}{2}\chi(M)$. Then

$$\Delta(G) \geq \deg(v) \geq \Delta(G) - 1 + |N_G(u) \cap N_G(v)|.$$
If \( \deg(v) = \Delta(G) - 1 \), then \( |N_G(u) \cap N_G(v)| = 0 \), this case is in the previous case. If \( \deg(v) = \Delta(G) \), then \( |N_G(u) \cap N_G(v)| = 1 \), and \( \gon(f) \geq 4 \) or \( \gon(f') \geq 4 \) holds. Then

\[
\frac{1}{\deg(u)} + \frac{1}{\deg(v)} + \frac{1}{\gon(f)} + \frac{1}{\gon(f')} - 1 - \frac{\chi(M)}{|E(G)|} \leq \frac{2}{12 - \chi(M)} + \frac{2}{12 - \chi(M)} + \frac{1}{3} + \frac{1}{4} - 1 - \frac{8\chi(M)}{(10 - \chi(M))(12 - \chi(M))} = \frac{12(10 - \chi(M)) - 5(10 - \chi(M))(12 - \chi(M)) - 96\chi(M)}{12(10 - \chi(M))(12 - \chi(M))} \leq \frac{-5\chi(M)^2 - 2\chi(M) - 480}{12(10 - \chi(M))(12 - \chi(M))} \leq \frac{-5(\chi(M) + 1/5)^2 - 2339/5}{12(10 - \chi(M))(12 - \chi(M))} < 0.
\]

If \( \deg(u) \geq 7 - \frac{1}{2}\chi(M) \) and \( \deg(v) \geq 7 - \frac{1}{2}\chi(M) \), then

\[
|E(G)| \geq \frac{1}{2} \left( 5 - \frac{\chi(M)}{2} \right) \left( 6 - \frac{\chi(M)}{2} \right) + \frac{1}{2} \cdot 2 \left( 7 - \frac{\chi(M)}{2} \right) = \frac{1}{8}(10 - \chi(M))(12 - \chi(M)) + 4(14 - \chi(M)) = \frac{1}{8}(\chi(M)^2 - 26\chi(M) + 176).
\]

In this case,

\[
\frac{1}{\deg(u)} + \frac{1}{\deg(v)} + \frac{1}{\gon(f)} + \frac{1}{\gon(f')} - 1 - \frac{\chi(M)}{|E(G)|} \leq \frac{2}{14 - \chi(M)} + \frac{2}{14 - \chi(M)} + \frac{1}{3} + \frac{1}{3} - \frac{8\chi(M)}{3(14 - \chi(M))(\chi(M)^2 - 26\chi(M) + 176) - 24\chi(M)(14 - \chi(M))} = \frac{\chi(M)^3 - 4\chi(M)^2 - 446\chi(M) - 352}{3(14 - \chi(M))(\chi(M)^2 - 26\chi(M) + 176)} < 0.
\]

Summing over all edges of \( G \) yields

\[
\sum_{e=uv \in E(G)} \left( \frac{1}{\deg(u)} + \frac{1}{\deg(v)} + \frac{1}{\gon(f_e)} + \frac{1}{\gon(f'_e)} - 1 - \frac{\chi(M)}{|E(G)|} \right) < 0,
\]

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which is a contradiction to Euler’s formula stating

\[
\sum_{uv \in E(G)} \left( \frac{1}{\deg(u)} + \frac{1}{\deg(v)} \right) = |V(G)|,
\]

\[
\sum_{e \in E(G)} \left( \frac{1}{\text{gon}(f_e)} + \frac{1}{\text{gon}(f'_e)} \right) = |F(G)|,
\]

and

\[
\chi(M) = |V(G)| - |E(G)| + |F(G)|
\]

\[
= \sum_{e=uv \in E(G)} \left( \frac{1}{\deg(u)} + \frac{1}{\deg(v)} - 1 + \frac{1}{\text{gon}(f_e)} + \frac{1}{\text{gon}(f'_e)} \right),
\]

\[
\sum_{e=uv \in E(G)} \left( \frac{1}{\deg(y)} + \frac{1}{\deg(v)} + \frac{1}{\text{gon}(f_e)} + \frac{1}{\text{gon}(f'_e)} - 1 - \frac{\chi(M)}{|E(G)|} \right) = 0.
\]

Therefore

\[
b_R(G) \leq 2\Delta(G) + 1 - \left\lfloor \frac{\chi(M)}{2} \right\rfloor.
\]

□

References


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Abstract

Let G be a simple graph, and its vertex sets is denoted by V(G). A set D ⊂ V(G) is the dominating set if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is the domination number γ(G). Clearly, for any spanning subgraph H of G, γ(H) ≥ γ(G). The bondage number of G, denoted by b(G), is the minimum cardinality of a set of edges B ⊂ E(G) such that γ(G − B) > γ(G), where G − B is the graph with V(G − B) = V(G) and E(G − B) = E(G) \ B.

A function f : V(G) → {0, 1, 2} is a Roman dominating function if every vertex v for which f(v) = 0 is adjacent to at least one vertex u for which f(u) = 2. The weight of a Roman dominating function is the value ∑v∈V(G) f(v). The Roman domination number of a graph G, denoted by γ_R(G), is the minimum weight of a Roman dominating function of G. The Roman bondage number b_R(G) of a graph G is the cardinality of a smallest set of edges B ⊂ E(G) for which γ_R(G − B) > γ_R(G), where V(G − B) = V(G) and E(G − B) = E(G) \ B.

In this paper, for a graph G on a closed surface M, we get an upper bound for the Roman bondage number b_R(G) of G by Euler characteristic χ(M) of M.