構造群の同型を誘導する領域交差変化及びその応用: リンク図形における応用

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Structures of homomorphisms induced by region crossing change on link diagram, and its application to region freeze crossing change

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1. Introduction

A link is a union of mutually disjoint, finite number of oriented 1-sphere(s) in the 3-sphere $S^3$ (Figure 1). In particular, one component link is called a knot. We say that a knot $K$ is trivial if $K$ bounds a disk in $S^3$. Let $S^2$ be a 2-sphere in $S^3$. Take and fix two points $n_\infty, s_\infty$ in $S^3$. Then $S^3 \setminus \{n_\infty, s_\infty\}$ is homeomorphic to $S^2 \times (0,1)$. We may suppose that $S^2$ corresponds to the level the 2-sphere $S^2_{f \infty 2}$ corresponds to the level the 2-sphere $S^2_{f \infty 2}$. Then let $p : S^3 \setminus \{n_\infty, s_\infty\} \to S^2$ be the map corresponding to the projection $S^2 \times (0,1) \to S^2 \times \{1/2\}$. We call $p$ a projection map. Further we suppose, unless otherwise specified, that each link $L$ is disjoint from $\{n_\infty, s_\infty\}$ (hence, $p(L) \subset S^2$ is defined). We call a point $c$ of the image $p(L)$ a multiple point if $|p^{-1}(c) \cap L| > 1$. We say that $c \in p(L)$ is an $n$-multiple point if $|p^{-1}(c) \cap L| = n$. A 2-multiple point is called a double point. In this paper, we assume that for each link $L$ the set of multiple points of $p(L)$ consists of finitely many transverse double points (see Figure 2). Each double point $c$ of $p(L)$ is called a crossing. Note that for a small disk neighborhood of a crossing $c$, denoted by $N(c)$, $p(L) \cap N(c)$ is formed by two arcs, say $a^+$ and $a^-$ such that $a^+ \cap a^- = c$. In order to distinguish upper/lower information of $a^+$ and $a^-$, we erase the arc which is lower than the other (see Figure 3). We call this figure a link diagram of $L$. In particular, if $L$ is a knot, then link diagrams of $L$ is called a knot diagram. Then $p(L)$ is called the projection of a link (knot) diagram.

Figure 1. links

Figure 2. transverse double point
In Knot Theory, operations on knot diagram are interesting research subjects. In particular, an operation is called an *unknotting operation*, if any knot diagram can be transformed into that of a trivial knot by a sequence of the operations and isotopies. For example, the operation called *crossing change* is an important unknotting operation. Let $D$ be a link diagram and $c$ a crossing of $D$. Then let $D_{cc}(c)$ be the link diagram obtained from $D$ by replacing $N(c)$ as in Figure 4. Then we say that the link diagram $D_{cc}(c)$ obtained from $D$ by crossing change at $c$. In [11], Hitoshi Murakami defined ♯-operation and proved that ♯-operation is an unknotting operation. Further, in [12], Hitoshi Murakami and Yasutaka Nakanishi defined ∆-operation and proved that ∆-operation is an unknotting operation. Later, Jim Hoste, Hitoshi Murakami and Kouki Taniyama defined $H(2)$-operation in [7] and proved that $H(2)$-operation is an unknotting operation (Figure 5).

In 2010, Kengo Kishimoto introduced a new operation on link diagram called region crossing change in a seminar at Osaka City University (see Section 1 in [13]), which is now studied by several authors (e.g. [1], [3], [10]). Moreover a game and a switching
system based on this concept was proposed by Akio Kawauchi, Kengo Kishimoto and Ayaka Shimizu [14](Figure 6).

Let $D$ be a link diagram of a link $L$ on the 2-sphere. Then $|D|$ denotes the projection $p(L)$ and this is regarded as a 4-valent plane graph. Each component of $S^2 \setminus |D|$ is called a region of $D$. For a region $R$ of $D$, $D_{rcc}(R)$ denotes the link diagram obtained from $D$ by changing the upper/lower information of the crossings on the boundary of $R$ (Figure 7). We say that $D_{rcc}(R)$ is obtained from $D$ by a region crossing change at $R$. In general, for a set $H$ of regions of $D$, the link diagram obtained from $D$ by region crossing change at $H$ is defined to be the link diagram obtained by the composition of region crossing changes at each element. It is easy to see that this is well-defined, that is, the obtained link diagram is independent from the orders of the elements of $H$.

For a convenience of the accessibility of $H$, we geometrically represent $H$ on the diagram $D$ by shading the regions which belong to $H$, and we call such representative a coloring of $D$.

It is natural to ask what kind of link diagrams are obtained by the region crossing changes on the given diagram. Then Ayaka Shimizu proved that region crossing change is an unknotting operation in [13]. In fact, she showed that any crossing of each knot diagram can be changed by a region crossing change. This result immediately implies the next theorem.
Theorem 1.1. For any knot diagram $D$ we have the following: for any set of crossings of $D$ there is a set of regions $H$ such that exactly the crossings are changed by the region crossing change at $H$.

On the other hand, it is remarked in [13] that 2-component trivial link diagram is not obtained from the link diagram of Figure 8 by any region crossing change. This phenomenon was studied by Zhiyun Cheng and Hongzhu Gao ([3]). They showed that region crossing change on $D$ induces a $\mathbb{Z}_2$-homomorphism (precise definition is given below) and made use of it to study region crossing change, and gave a necessary and sufficient condition for the given link to be transformed to a trivial link by region crossing change and isotopies (Theorem 1.3 in [3]).

![Figure 8. Hopf link]

Recently, in [8], Ayumu Inoue and Ryo Shimizu introduced another operation of link diagram called region freeze crossing change, as a mutant of region crossing change. In particular, they showed that there exists a knot diagram such that some changes of crossings can not be realized by region freeze crossing change. Moreover, they gave a necessary and sufficient condition for the exchangeability of a crossing of the knot diagram by region freeze crossing change (Theorems 3.4 and 3.5 in [8]).

In this paper we study the following two topics.

**Topic 1:** To analyze the homomorphism introduced by Cheng and Gao ([3]) for studying region crossing change.

**Topic 2:** To apply the idea of homomorphism induced by region crossing change to study region freeze crossing change, and give a generalization of the result of Inoue-Shimizu’s for link diagram.

Each topic is studied as follows. For the statement of the results, we settle some settings.

In general, for a set $X$, $2^X$ denotes the power set of $X$. We note that $2^X$ admits a $\mathbb{Z}_2$-linear structure by using symmetric difference as the sum. Let $C(D)$ (resp. $R(D)$) be the set of the crossings (resp. regions) of $D$. Then Cheng and Gao in [3] claimed that the region crossing change on $D$ induces a $\mathbb{Z}_2$-linear map $\Phi : 2^{R(D)} \to 2^{C(D)}$, and made use of it to study region crossing change. Particularly, they showed:

**Fact 1** ([Theorem 1.4 [3]]) Let $D$, $\Phi$ be as above. Then $\dim(\text{Im}\Phi) = q - n + 1$, where $q$ is the number of the crossings, and $n$ is the number of components of the link represented.
We note that if \( n = 1 \) (i.e., \( D \) is a knot diagram), then Fact 1 implies \( \dim(\text{Im}\Phi) = q = \dim(\mathcal{C}(D)) \), hence \( \text{Im}\Phi = 2^\mathcal{C}(D) \). This gives an alternative expression of Theorem 1.1.

**Remark 1.1.** By Euler characteristic arguments, it is easy to see that Fact 1 implies (see Subsection 2.4):

\[
(1) \quad \dim(\ker\Phi) = n + 1.
\]

In Subsection 3.1, we study the kernel of \( \Phi \). In fact, we show that if \( D \) is irreducible (for the definitions, see Section 2.1), then \( \ker\Phi \) admits a neat basis. The precise statement is as follows.

We say that a coloring of \( D \) is a **checker board coloring** of \( D \) if the following condition is satisfied.

For any point \( p \) on \( |D| \setminus \{\text{crossings}\} \), a small disk neighborhood of \( p \) is divided into two parts by \( |D| \). Then one part is shaded and the other is non-shaded as in Figure 9.

![Figure 9](image)

Let \( D \) be a link diagram and \( k_1, \ldots, k_n \) the knot diagrams corresponding to the components of \( D \). We say that a coloring of \( D \) is a **componentwise checkerboard coloring associated with** \( k_i \), if it gives a checkerboard coloring of \( k_i \) by ignoring the components other than \( k_i \) (see Definition 3.1). We note that for each \( k_i \), there are two checkerboard colorings associated with \( k_i \). Take and fix a point \( \infty \in S^2 \setminus |D| \). Then \( B_i (i = 1, \ldots, n) \) denotes the componentwise checkerboard coloring associated with \( k_i \) such that the region containing \( \infty \) is not shaded. For the sake of simplicity, we denote the set of the regions \( \mathcal{R}(D) \) by \( B_0 \).

Then we have:

**Theorem 1.2** ([5]). Let \( D = k_1 \cup \cdots \cup k_n, \mathcal{R}(D), \mathcal{C}(D), \Phi : 2^{\mathcal{R}(D)} \to 2^{\mathcal{C}(D)}, B_0, B_1, \ldots, B_n \) be as above. Suppose that \( D \) is irreducible. Then, \( \{B_0, B_1, \ldots, B_n\} \) is a basis of \( \ker\Phi \).
Next, we study the image of $\Phi$ and the cokernel of $\Phi$. In fact, we give a generating system of $\text{Im}\Phi$ and a representative of $\text{Coker}\Phi$. The precise statements are as follows.

**Notations.**

Let $D = k_1 \cup \cdots \cup k_n$ be as above. Let $C_{ij}$ ($i, j = 1, \ldots, n$) be the set of crossings each of which formed by subarcs of $k_i$ and $k_j$. For an integer $s \geq 2$, a set of mutually different $s$ crossings $\{c_1, \ldots, c_s\} \subset C(D)$ is said to be a $v$-cycle of length $s$, if there exists $\{p_1, p_2, \ldots, p_s\} \subset \{1, 2, \ldots, n\}$ ($p_i \neq p_u$, for each $t \neq u$) such that $c_1 \in C_{p_1p_2}$, $c_2 \in C_{p_2p_3}, \ldots, c_{s-1} \in C_{p_{s-1}p_s}$, $c_s \in C_{p_sp_1}$.

Then $\mathcal{T}^s(\subset 2^{C(D)})$ denotes the set consisting of the $v$-cycles of length $s$. Further, for each $c \in C(k_i) (= C_{ii})$ ($i = 1, \ldots, n$), $\{c\}$ is called a $v$-cycle of length $1$, and $\mathcal{T}^1$ denotes the set of the $v$-cycles of length $1$. Let $H$ be the subspace of $2^{C(D)}$ generated by $\mathcal{T}^1 \cup \mathcal{T}^2 \cup \cdots \cup \mathcal{T}^n$. Then we show:

**Theorem 1.3.** Let $D, \Phi, H$ be as above. Then

$$H = \text{Im}\Phi.$$  

Next we discuss about Coker$\Phi$. Let $D = k_1 \cup \cdots \cup k_n$, $C_{ij}$ be as above. Suppose that $D$ is non-separable. Let $G_D$ be the graph obtained from $D$ as follows. Each vertex $v_i$ ($i = 1, \ldots, n$) of $G_D$ corresponds to the component $k_i$ of $D$ and there is an edge between $v_i$ and $v_j$ ($i \neq j$) if and only if $C_{ij} \neq \emptyset$. Then let $T_D$ be a spanning tree of $G_D$, i.e., $T_D$ is a subtree of $G_D$ such that the set of vertices of $T_D$ equals to the set of vertices of $G_D$. Let $\{e_1, \ldots, e_{n-1}\}$ be the set of the edges of $T_D$. Fix a crossing $d_i \in C(D)$ corresponding to $e_i$ ($i = 1, \ldots, n-1$) (Figure 10).

**Theorem 1.4.** Let $D, \Phi, d_i$ ($i = 1, \ldots, n-1$) be as above. We regard $2^{C(D)}$ as a group whose operation is given by symmetric difference. Then, the coset decomposition of $2^{C(D)}$ by $\text{Im}\Phi$ has the following presentation;

$$2^{C(D)} = \bigoplus_{Y \in 2^{\{d_1, \ldots, d_{n-1}\}}} (Y + \text{Im}\Phi).$$

Theorems 1.3, 1.4 enable us to consider a decomposition of the set of the link diagrams with the same projection, where two link diagrams belong to the same subset if and only if they are mutually transformed by a region crossing change. For example, for the projection $P$ in Figure 11, there are 4 link diagrams each of which has $P$ as the projection. We can show that (for details, see Example 3.1) they are decomposed into 2 sets depicted in Figure 12.

For Topic 2, we start with reviewing the result of Inoue-Shimizu on region freeze crossing change. Let $D$ be a link diagram. Recall that $C(D)$ (resp. $R(D)$) denotes the set of crossings (resp. regions) of $D$. For a region $R \in R(D)$, let $C(D)_{\partial R} = \{c \in C(D) \mid$
Then, $D_{rfcc}(R)$ denotes the link diagram obtained from $D$ by changing the upper/lower information of $\mathcal{C}(D) \setminus \mathcal{C}(D)_{\partial R}$ (Figure 13). In general, for a set $H(\subseteq \mathcal{R}(D))$ of regions of $D$, the link diagram obtained from $D$ by region freeze crossing change at $H$ is defined to be the link diagram obtained by the composition of the region freeze crossing changes of the elements of $H$. We will see that this is also well-defined (see Proposition 2.1 and Proposition 2.2).

Then, in [8], Inoue-Shimizu showed that there exists a knot diagram such that some changes of crossings can not be realized by region freeze crossing change (Figure 14). With using the homomorphism $\Phi$, their result (Theorems 3.4 and 3.5 in [8]) can be rephrased as follows;
Theorem 1.5. Let $D$ be a knot diagram. Let $\mathcal{R}(D)$, $\mathcal{C}(D)$, $\Phi$ be as above. Then any crossing of $D$ can be changed by region freeze crossing change if and only if one of the following conditions is satisfied.

1. there exists an element of $\ker \Phi$ consisting of an odd number of regions.
2. the set of such crossing $c$ of $D$ that satisfies the following condition ($\ast$) consists of even number of elements.

\begin{align*}
(\ast) & \quad \text{there exists } J_c \in 2^{\mathcal{R}(D)} \text{ such that } \Phi(J_c) = \{c\} \text{ and } J_c \text{ consists of an odd number of regions.}
\end{align*}

In this paper, we give a result which is similar to Theorem 1.5 for links. Let $H = \{R_\alpha\}_{\alpha \in \Lambda}$ be a set of regions of a link diagram $D$. For each crossing $c$, $d_c(H)$ denotes the number of the elements $R_\alpha$ of $H$ such that $c \in \partial R_\alpha$. Next let $\Psi: 2^{\mathcal{R}(D)} \to 2^{\mathcal{C}(D)}$ be the map defined as follows:

$$
\Psi(H) = \begin{cases} 
\{c \mid d_c(H) \equiv 1 \pmod{2}\} & \text{if } |H| \equiv 0 \pmod{2} \\
\{c \mid d_c(H) \equiv 0 \pmod{2}\} & \text{if } |H| \equiv 1 \pmod{2}
\end{cases}
$$

We show that $\Psi$ is a homomorphism (Proposition 2.1 and Assertion 2). Then we show that $\Psi(H)$ corresponds to the set of crossings changed by region freeze crossing change at $H$ (Proposition 2.2).

Let $\mathcal{M} = \{M \in 2^{\mathcal{R}(D)} \mid \Phi(M) = \mathcal{C}(D)\}$. We note that some results in [3] implies that $\mathcal{M} \neq \emptyset$ (see Proposition 4.1). Let $\mathcal{H}_2 = \{H \in \ker \Phi \mid |H| \equiv 0 \pmod{2}\}$ and
\( \mathcal{M}_1 = \{ M \in \mathcal{M} \mid |M| \equiv 1 \text{ mod } 2 \} \). We say that \( \Phi \) is even type, if \( \ker \Phi = \mathcal{H}_2 \), i.e., each element of \( \ker \Phi \) consists of even number of regions. Then we have:

**Theorem 1.6.** Let \( D, \Phi, \Psi, \mathcal{H}_2, \mathcal{M}, \mathcal{M}_1 \) be as above. Then we have the following.

(i) If \( \Phi \) is even type (i.e., \( \ker \Phi = \mathcal{H}_2 \)), then we have the following.

(a) If \( \mathcal{M}_1 = \emptyset \), then \( \ker \Psi = \ker \Phi \) (i.e., \( \mathcal{H}_2 \)). In particular, \( \dim(\ker \Psi) = n + 1 \).

(b) If \( \mathcal{M}_1 \neq \emptyset \), then for any \( M \in \mathcal{M}_1 \), \( \ker \Psi = (\ker \Phi) \Pi (M + \ker \Phi) \) (i.e., \( \mathcal{H}_2 \Pi (M + \mathcal{H}_2) \)). In particular, \( \dim(\ker \Psi) = n + 2 \).

(ii) If \( \Phi \) is not even type, then (1) \( |\mathcal{H}_2| = \frac{1}{2} |\ker \Phi| \) (hence, \( \dim(\mathcal{H}_2) = n \)), and;

(2) \( \mathcal{M}_1 \neq \emptyset \) and for any \( M \in \mathcal{M}_1 \), \( \ker \Psi = \mathcal{H}_2 \Pi (M + \mathcal{H}_2) \). In particular, \( \dim(\ker \Psi) = n + 1 \).

Figure 15 depicts the link diagrams satisfying the conditions of the conclusions of Theorem 1.6.

![Figure 15](image)

**Remark 1.2.** We note that in the conclusion (i)-(b) of Theorem 1.6, we have \( \mathcal{M} = \mathcal{M}_1 \) (see Lemma 4.6). Hence the condition: “\( \Phi \) is even type and \( \mathcal{M}_1 \neq \emptyset \), i.e., \( \exists M \in \mathcal{M} \) s.t. \( |M| \equiv 1 \text{ mod } 2 \)” is equivalent to the condition “\( \Phi \) is even type and \( \mathcal{M}_1 = \mathcal{M} \), i.e., \( \forall M \in \mathcal{M}, |M| \equiv 1 \text{ mod } 2 \)”.

Note that in the conclusions of Theorem 1.6, \( \dim(\ker \Psi) = n + 2 \) holds if and only if the condition of (i)-(b) holds. This fact shows:

**Corollary 1.1.** Let \( D, \mathcal{M}, \Phi, \Psi \) be as above. Then \( \dim(\ker \Psi) = n + 1 \) or \( n + 2 \). We have \( \dim(\ker \Psi) = n + 2 \) if and only if \( \Phi \) is even type and \( \exists M \in \mathcal{M} \) such that \( |M| \equiv 1 \text{ mod } 2 \).

We note that in Section 4, we show \( \text{Im}\Psi \subset \text{Im}\Phi \) (Remark 4.1). Since \( \dim(\ker \Phi) = n + 1 \) (see Remark 1.1), this fact together with Theorem 1.6 shows that \( \text{Im}\Psi = \text{Im}\Phi \) if and only if either conclusion (i)-(a) or (ii) holds. Further Corollary 1.1 and Remark 1.1 show:

**Corollary 1.2.** Let \( D, \mathcal{R}(D), \Phi, \Psi \) be as above. Then \( \text{Im}\Psi \subset \text{Im}\Phi \) if and only if \( \Phi \) is even type and \( \exists M \in \mathcal{M} \) s.t. \( |M| \equiv 1 \text{ mod } 2 \).
Then it is natural to ask, in the situation of Corollary 1.2, for a given $J \subseteq \operatorname{Im}\Phi$, whether $J \subseteq \operatorname{Im}\Psi$ or not. Then we show:

**Theorem 1.7.** Let $D$, $\mathcal{R}(D)$, $\Phi$, $\Psi$ be as above. Suppose that $\operatorname{Im}\Psi \subsetneq \operatorname{Im}\Phi$. Let $J \subseteq \operatorname{Im}\Phi$. Then $J \not\subseteq \operatorname{Im}\Psi$ if and only if $\exists H \in 2^{\mathcal{R}(D)}$ s.t. $\Phi(H) = J$ and $|H| \equiv 1 \ mod \ 2$.

This paper is organized as follows. Section 2 is the preliminaries. In Subsection 2.1, we quickly review some fundamental concepts (link, link diagram, region, graph, cycle etc.) in knot and link theory and graph theory. In Subsection 2.2, we introduce a matrix called incidence matrix obtained from a link diagram defined by Cheng and Gao, which will be used in the proofs of results in this paper. In Subsection 2.3, we show that a projection $|D|$ of a link diagram $D$ can be regarded as a 4-valent plane graph. Then we can identify the set of the regions $\mathcal{R}(D)$ with the set of the faces $F(|D|)$, and the set of the crossings $\mathcal{C}(D)$ with the set of the vertices $V(|D|)$. We reformulate the map $\Phi : 2^{\mathcal{R}(D)} \to 2^{\mathcal{C}(D)}$ as $\varphi : 2^{F(|D|)} \to 2^{V(|D|)}$. In Subsection 2.4, we show that the map $\varphi$ corresponds to region crossing change. In Subsection 2.5, we introduce a map $\psi : 2^{F(|D|)} \to 2^{V(|D|)}$, and show that it corresponds to region freeze crossing change. In Section 3, we study the structure of the map $\varphi$. In Subsection 3.1, we study $\ker \varphi$. In fact, we give Theorem 1.2' corresponding to Theorem 1.2, and give a proof of the theorem. In Subsection 3.2, we study $\operatorname{Im} \varphi$. Then we give Theorem 1.3' corresponding to Theorem 1.3. In Subsection 3.3, we study $\operatorname{Coker} \varphi$. Then we give Theorem 1.4' corresponding to Theorem 1.4. In Subsection 3.3, we give proofs of Theorems 1.3' and 1.4', and these immediately give proofs of Theorems 1.3 and 1.4. In Section 4, we study the structure of $\psi$. In fact, we give Theorems 1.6' and 1.7', which correspond to Theorems 1.6 and 1.7, and we prove Theorems 1.6' and 1.7'.
2. Preliminaries

In this paper, we work in the smooth category.

2.1. Terminologies in Knot and Link Theory and Graph Theory. In this subsection, we introduce some terminologies. A link is a union of mutually disjoint, finite number of oriented 1-sphere(s) in the 3-sphere $S^3$. In particular, one component link is called a knot. We say that a knot $K$ is trivial if $K$ bounds a disk in $S^3$. Let $S^2$ be a 2-sphere in $S^3$. Take and fix two points $n_{\infty}, s_{\infty}$ in $S^3$. Then $S^3 \setminus \{n_{\infty}, s_{\infty}\}$ is homeomorphic to $S^2 \times (0, 1)$. We may suppose that $S^2$ corresponds to the level 2-sphere $S^2 \times \{\frac{1}{2}\}$. Then let $p : S^3 \setminus \{n_{\infty}, s_{\infty}\} \to S^2$ be the map corresponding to the projection $S^2 \times (0, 1) \to S^2 \times \{\frac{1}{2}\}$. We call $p$ a projection map. Further we suppose, unless otherwise specified, that each link $L$ is disjoint from $\{n_{\infty}, s_{\infty}\}$ (hence, $p(L) \subset S^2$ is defined). We call a point $c$ of the image $p(L)$ multiple point if $p^{-1}(c) \cap L$ contains more than one points. We say that $c (\in p(L))$ is an $n$-multiple point if $|p^{-1}(c) \cap L| = n$. A 2-multiple point is called a double point. In this paper, we assume that for each link $L$ the set of multiple points of $p(L)$ consist of finitely many transverse double points (see Figure 16).

![Figure 16](image_url)

Figure 16. If $p$ is regular, then $p(L)$ contains double points like (a) and, (b), (c) and (d) are not allowed.

Each double point $c$ of $p(L)$ is called a crossing. Note that for a small disk neighborhood of a crossing $c$, denoted by $N(c)$, $p(L) \cap N(c)$ is formed by two arcs, say $a^+$ and $a^-$, such that $a^+ \cap a^- = c$. In order to distinguish upper/lower information of $a^+$ and $a^-$, we erase the arc which is lower than the other (see Figure 17). We call this figure $D$ a link diagram of $L$. In particular, if $L$ is a knot, then $D$ is called a knot diagram. Then $p(L)$ is called the projection of $D$, and $p(L)$ is denoted by $|D|$. Each component of $S^2 \setminus |D|$ is called a region of $D$.

We say that a link diagram $D$ is separable if there exists a circle $E$ on the 2-sphere $S^2$ such that $E \cap |D| = \emptyset$ and the regions separated by $E$ contain a component of $|D|$. Let $c$ be a crossing of $D$. The crossing $c$ is reducible if there exists a circle $E$ on $S^2$ which transversely intersects $|D|$ only in $c$. We say that $D$ is reducible if $D$ has a reducible crossing. We say that $D$ is irreducible if it contains no reducible crossing.

For the definitions of other standard terms in knot and link theory, we refer to [9].
Next we review some basic definitions and terminologies in Graph Theory. A graph $G$ is a pair of sets $(V, E)$ where $V$ is nonempty, and $E$ is a (possibly empty) set of unordered pairs of elements of $V$. The elements of $V$ are called the vertices of $G$ and the elements of $E$ are called the edges of $G$. If $e \in E$ represents the pair $(x, y)$, then we say that $e$ is an edge between $x$ and $y$. Then we shall denote such an edge by $xy$. When it is needed to specify $V$ (resp. $E$) is the set of vertices (resp. edges) of $G$, we may use notations $V(G)$ for $V$ and $E(G)$ for $E$. Usually we represent the vertices by points in the plane, and an edge by a segment connecting two vertices in the plane. A graph $G$ is a plane graph if $G$ is realized in the plane (or $S^2$) with no self intersection. The segments might be straight or curved (see Figure 18). Let $x$ and $y$ be vertices of $G$. We say that $x$ is adjacent to $y$ if there is an edge $e$ between $x$ and $y$. Then we say that $e$ is incident with $x$. We also say that $x$ is incident with $e$. If an edge is incident with only one vertex, then the edge is called a loop (Figure 18). The degree of a vertex $v$ in $G$ is the number of edges incident with $v$, where loops are counted twice.

A subgraph of a graph $G = (V(G), E(G))$ is a graph $H$ such that every vertex of $H$ is a vertex of $G$, and every edge is an edge of $G$ also. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph with $n + 1$ vertices $\{x_0, x_1, \ldots, x_{n-1}, x_n\}$ (possibly, $x_i = x_j$ for $i \neq j$) and edges $\{x_0x_1, x_1x_2, \ldots, x_{n-1}x_n\}$ is called a path of length $n$. We call $x_0$ and $x_n$ the end vertices of the path, and we say that the vertices $x_0$ and $x_n$ are connected by the path. The graph with $n$ vertices $\{x_0, x_1, \ldots, x_{n-1}\}$ (possibly, $x_i = x_j$ for $i \neq j$) and edges
\{x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, x_{n-1}x_0\} is called a cycle of length \(n\). A graph \(G\) is connected if for any two vertices \(a\) and \(b\) there is a subgraph of \(G\) which is a path connecting \(a\) and \(b\). A tree is a connected graph that contains no subgraph which is a cycle. A subgraph \(H\) of \(G\) is called a spanning subgraph if the set of the vertices of \(H\) is the set of the vertices of \(G\), i.e., \(V(H) = V(G)\). A spanning tree of \(G\) is a spanning subgraph of \(G\) that is a tree.

For definitions of other standard terms in graph theory, we refer to [4].

2.2. Incidence matrix. Let \(D\) be a link diagram. Let \(\mathcal{C}(D) = \{c_1, \ldots, c_q\}\) and \(\mathcal{R}(D) = \{R_1, \ldots, R_l\}\) be the set of the crossings and the regions. Then we induce a matrix \(A(D)\), each \((i,j)\)-entry \(m_{ij}\) of which is defined by:

\[
m_{ij} = \begin{cases} 
1 & \text{if } c_j \in \partial R_i \\
0 & \text{if } c_j \notin \partial R_i
\end{cases}
\]

We note that \(A(D)\) is the same as the incidence matrix defined by Cheng and Gao in [3] up to permutations of rows and columns. Then in Theorem 1.4 of [3] (see, also [2]) the following is shown.

**Theorem 2.1.** Let \(D, A(D)\) be as above. Suppose that \(D\) is non-separable. Then we have

\[
n = q - \text{rank}_{\mathbb{Z}_2}A(D) + 1,
\]

where \(n\) is the number of the components of the link represented by \(D\) and \(q\) is the number of the crossings of \(D\).

It is easy to generalize Theorem 2.1 to separable \(D\). In fact, the next holds.

**Claim 1.** Let \(D, A(D), n, q\) be as above. Then we have

\[
(2) \quad n = q - \text{rank}_{\mathbb{Z}_2}A(D) + c,
\]

where \(c\) is the number of connected components of \(|D|\).

2.3. Representing a projection of a link as a plane graph. A graph \(G\) is called a 4-valent graph if the degree of each vertex is 4. Let \(\tilde{G} (\subset S^2)\) be a union of mutually disjoint 4-valent connected plane graph(s) or circle(s) denoted by \(\tilde{G}_c^1, \tilde{G}_c^2, \ldots, \tilde{G}_c^m\). Recall that for a graph \(G\), \(V(G)\) (resp. \(E(G)\)) denotes the set of the vertices of \(G\) (resp. edges of \(G\)). We extend this notation for a circle \(G\) so that \(V(G) = \emptyset, E(G) = \emptyset\). Then we define \(V(\tilde{G}) = V(\tilde{G}_1^c) \cup \cdots \cup V(\tilde{G}_m^c), \) and \(E(\tilde{G}) = E(\tilde{G}_1^c) \cup \cdots \cup E(\tilde{G}_m^c)\). Note that in a neighborhood of each vertex, \(\tilde{G}\) looks as the union of two arcs intersecting transversely in one point. This shows that we may regard \(\tilde{G}\) as the union of the image(s) of immersion(s) of a circle. Hence we may regard \(\tilde{G}\) as a projection of a link. Then we see that \(\tilde{G}\) admits an expression \(\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_n\) where each \(\tilde{G}_i\) represents a component of the link. Note
that $\tilde{G}_i$ represents a circuit in the graph $\tilde{G}$ or a cicle. With adopting a terminology in graph theory, we call each $\tilde{G}_i$ ($i = 1, \ldots, n$) a cut-through circuit of $\tilde{G}$. We note that for any subset of the cut-through circuits, say $\{\tilde{G}_{i_1}, \ldots, \tilde{G}_{i_k}\}$, $\tilde{G}_{i_1} \cup \cdots \cup \tilde{G}_{i_k}$ is a union of mutually disjoint 4-valent plane graph(s) or circle(s), hence the notation $V(\tilde{G}_{i_1} \cup \cdots \cup \tilde{G}_{i_k})$ makes sense. In this paper, we sometimes regard a subset of $V(\tilde{G})$ as a subspace of $S^2$. Each component of $S^2 \setminus \tilde{G}$ is called a face of $\tilde{G}$. Note that a face may not be simply connected. Let $F(\tilde{G})$ be the set of the faces of $\tilde{G}$. Now we consider the power sets $2^{F(\tilde{G})}$ and $2^{V(\tilde{G})}$. For a convenience of the accessibility of $H \in 2^{F(\tilde{G})}$, we geometrically represent $H$ on the graph $\tilde{G}$ by shading the faces which belong to $H$, and we call such representative a coloring of $\tilde{G}$ as we did for link diagram. We introduce an addition on $2^{F(\tilde{G})}$ and $2^{V(\tilde{G})}$ by symmetric difference, that is, if $A$ and $B$ are elements of $2^{F(\tilde{G})}$ (resp. $2^{V(\tilde{G})}$), then the addition of $A$ and $B$ denoted by $A + B$ is given by $A + B = (A \setminus B) \cup (B \setminus A)$. Furthermore scalar multiplication by $\mathbb{Z}_2$ is defined by $0 \cdot A = \emptyset$, $1 \cdot A = A$ for each $A \in 2^{F(\tilde{G})}$ (resp. $2^{V(\tilde{G})}$). It is easy to show that these structures make $2^{F(\tilde{G})}$ (resp. $2^{V(\tilde{G})}$) a $\mathbb{Z}_2$-linear space.

**Remark 2.1.** It is easy to see that the following holds;

Let $X, +$ be as above. Suppose that $|X| < \infty$. Then for any $A_1, A_2 \in 2^X$, we have: $|A_1 + A_2| \equiv |A_1| + |A_2| \mod 2$.

2.4. **Homomorphism induced by region crossing change.** Let $\tilde{G}, V(\tilde{G}), F(\tilde{G})$ be as in Subsection 2.3. Then let $H = \{F_\alpha\}_{\alpha \in A}$ be a set of faces of $\tilde{G}$. For each vertex $v$, $\tilde{d}_v(H)$ denotes the number of the elements $F_\alpha$ such that $v \in \partial F_\alpha$. We define a map $\varphi$ from $2^{F(\tilde{G})}$ to $2^{V(\tilde{G})}$ as follows.

For $H(\in 2^{F(\tilde{G})})$, $\varphi(H) := \{v \mid \tilde{d}_v(H) \equiv 1 \mod 2\}$.

**Assertion 1** The map $\varphi$ is a homomorphism.

*Proof.* If $\tilde{d}_v(H_1 + H_2) \equiv 1 \mod 2$, then by Remark 2.1 either “$\tilde{d}_v(H_1) \equiv 1 \mod 2$ and $\tilde{d}_v(H_2) \equiv 0 \mod 2$” or “$\tilde{d}_v(H_1) \equiv 0 \mod 2$ and $\tilde{d}_v(H_2) \equiv 1 \mod 2$”. Hence,

$v \in \varphi(H_1 + H_2)$

$\iff v \in \{w \mid \tilde{d}_w(H_1 + H_2) \equiv 1 \mod 2\}$

$\iff v \in \{w \mid \tilde{d}_w(H_1) \equiv 1 \mod 2, \tilde{d}_w(H_2) \equiv 0 \mod 2\} \cup \{w \mid \tilde{d}_w(H_1) \equiv 0 \mod 2, \tilde{d}_w(H_2) \equiv 1 \mod 2\}$.

On the other hand,

$v \in \varphi(H_1) + \varphi(H_2)$

$\iff v \in \{w \mid \tilde{d}_w(H_1) \equiv 1 \mod 2\} + \{w \mid \tilde{d}_w(H_2) \equiv 1 \mod 2\}$

$\iff v \in (\{w \mid \tilde{d}_w(H_1) \equiv 1 \mod 2\} \setminus \{w \mid \tilde{d}_w(H_2) \equiv 1 \mod 2\}) \cup (\{w \mid \tilde{d}_w(H_1) \equiv 1 \mod 2\} \setminus \{w \mid \tilde{d}_w(H_2) \equiv 1 \mod 2\})$.
\[ v \in \{ w \mid d_w(H_1) \equiv 1 \mod 2, d_w(H_2) \equiv 0 \mod 2 \} \cap \{ w \mid d_w(H_1) \equiv 0 \mod 2, d_w(H_2) \equiv 1 \mod 2 \}. \]

These show that \( \varphi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2) \).

Let \( D \) be a link diagram such that \( |D| = \tilde{G} \). Then we can identify \( \mathcal{R}(D) \) with \( F(\tilde{G}) \), and \( \mathcal{C}(D) \) with \( V(\tilde{G}) \). For any set of regions \( H' = \{ R_{i1}, \ldots, R_{is} \} \), \( \delta_D(H') \) denotes the set of crossings changed by region crossing change at \( H' \). Then by the definition of region crossing change, we have

\[
\delta_D(H') = \delta_D(R_{i1} + \cdots + R_{is}) = \delta_D(R_{i1}) + \cdots + \delta_D(R_{is}).
\]

Then let \( F_{ij} \) \((j = 1, \ldots, s)\) be the face of \( \tilde{G} \) corresponding to \( R_{ij} \). By Assertion 1, for \( H := \{ F_{i1}, \ldots, F_{is} \} \in 2^{F(\tilde{G})} \),

\[
\varphi(H) = \varphi(F_{i1} + \cdots + F_{is}) = \varphi(F_{i1}) + \cdots + \varphi(F_{is}).
\]

Further for each \( F_j \), it is easy to show that \( \varphi(F_j) \) corresponds to \( \delta_D(R_j) \). These show:

**Claim 2.** If \( |D| \) is regarded as a 4-valent plane graph, where the set of regions \( H' \) of \( D \) corresponds to the set of faces \( H \) of \( |D| \), then \( \delta_D(H') \) corresponds to \( \varphi(H) \), i.e., \( v \) is changed by region crossing change at \( H' \) if and only if \( v \in \varphi(H) \).

**Definition 2.1.** For a connected 4-valent plane graph \( G \), if there is a circle \( E(\subset S^2) \) such that \( E \) intersects \( G \) in only one vertex \( v \) and for each component \( D \) of \( S^2 \setminus E \), we have \( D \cap G \neq \emptyset \), \( G \) is called reducible, where the vertex \( v \) is called a reducible vertex. Let \( \tilde{G} \) be as above. We say that \( \tilde{G} \) is reducible if a component of \( \tilde{G} \) is reducible. We say that \( \tilde{G} \) is irreducible if it is not reducible.

Then by the definition of \( \varphi \), we have the following.

**Observation 2.1.** Let \( p \in V(\tilde{G}) \). If \( p \) is not a reducible vertex, then \( p \in \varphi(H) \) if and only if a neighborhood of \( p \) looks as one of the figures in Figure 19. Furthermore, in the case when \( p \) is a reducible vertex (as in Figure 20), \( p \in \varphi(H) \) if and only if a neighborhood of \( p \) looks as one of the figures in Figure 20.

![Figure 19](image-url)
Observation 2.2. Let \( p \in V(\tilde{G}) \). If \( p \) is not a reducible vertex, then \( p \not\in \varphi(H) \) if and only if a neighborhood of \( p \) looks as one of the figures in Figure 21. Furthermore, in the case when \( p \) is a reducible vertex (as in Figure 22), \( p \in \varphi(H) \) if and only if a neighborhood of \( p \) looks as one of the figures in Figure 22.

A representative of \( \varphi \)
Let \( \tilde{G} \) and \( \varphi \) be as above. Let \( D \) be a link diagram such that \(|D| = \tilde{G}\). Then let \( F(\tilde{G}) := \{F_1, \ldots, F_p\}, V(\tilde{G}) := \{v_1, \ldots, v_q\} \). Let \( f_{F(\tilde{G})} : 2^{F(\tilde{G})} \rightarrow \mathbb{Z}_2^p \) be the isomorphism given by:

\[
\begin{align*}
    f_{F(\tilde{G})}(\{F_1\}) &= e_1 = (1, 0, \ldots, 0) \in \mathbb{Z}_2^p \\
    f_{F(\tilde{G})}(\{F_2\}) &= e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{Z}_2^p \\
    &\vdots \\
    f_{F(\tilde{G})}(\{F_p\}) &= e_p = (0, \ldots, 0, 1) \in \mathbb{Z}_2^p.
\end{align*}
\]
Let \( f_{V(\tilde{G})} : 2^{V(\tilde{G})} \to \mathbb{Z}_2^q \) be the isomorphism given by:
\[
\begin{align*}
f_{V(\tilde{G})}(\{v_1\}) &= f_1 = (1, 0, \cdots, 0) \in \mathbb{Z}_2^q, \\
f_{V(\tilde{G})}(\{v_2\}) &= f_2 = (0, 1, 0\cdots, 0) \in \mathbb{Z}_2^q, \\
&\vdots \\
f_{V(\tilde{G})}(\{v_q\}) &= f_q = (0, \cdots, 0, 1) \in \mathbb{Z}_2^q.
\end{align*}
\]

Then \( \tilde{\varphi} : \mathbb{Z}_2^p \to \mathbb{Z}_2^q \) denotes the homomorphism \( f_{V(\tilde{G})} \circ \varphi \circ (f_{F(\tilde{G})})^{-1} \). It is easy to see from the definition of \( A(D) \) that \( A(D) \) represents the \( \mathbb{Z}_2 \)-linear map \( \tilde{\varphi} \) with respect to the bases \( \{e_1, \ldots, e_p\} \) and \( \{f_1, \ldots, f_q\} \) (i.e. for each \( x \in \mathbb{Z}_2^p \), \( \tilde{\varphi}(x) = xA(D) \)). Then let \( c \) be the number of the connected components of \( |D| \). By using Euler characteristic, we see that \( p = q + c + 1 \). By Homomorphism Theorem in linear algebra this fact together with Claim 1 in Subsection 2.2 implies
\[
\dim(\ker \varphi) = \dim(2^{F(\tilde{G})}) - \dim(\text{Im} \varphi) = p - \text{rank}_{\mathbb{Z}_2} A(D) = (q + c + 1) - \text{rank}_{\mathbb{Z}_2} A(D) = (q + c + 1) - (q - n + c) = n + 1,
\]
where \( n \) is the number of the cut-through circuits of \( \tilde{G} \) (see Remark 1.1) or, equivalently, the number of the components of the link represented by \( D \).

2.5. **Homomorphism induced by region freeze crossing change.** Let \( \tilde{G}, F(\tilde{G}), V(\tilde{G}) \) be as in Subsection 2.3. Let \( D \) be a link diagram such that \( |D| = \tilde{G} \). Let \( \varphi : 2^{F(\tilde{G})} \to 2^{V(\tilde{G})} \) be as in Subsection 2.4. For a vertex \( v \) and a set of faces \( H \), let \( \tilde{\delta}_v(H) \) be as in Subsection 2.4. Further, let \( \psi : 2^{F(\tilde{G})} \to 2^{V(\tilde{G})} \) be the map defined as follows;
For \( H(\in 2^{F(\tilde{G})}) \),
\[
\psi(H) = \begin{cases} 
\{v \mid \tilde{\delta}_v(H) \equiv 1 \mod 2\} & \text{if } |H| \equiv 0 \mod 2 \\
\{v \mid \tilde{\delta}_v(H) \equiv 0 \mod 2\} & \text{if } |H| \equiv 1 \mod 2
\end{cases}
\]
By comparing the definitions of \( \varphi \) and \( \psi \), we have;

**Proposition 2.1.** For \( H \in 2^{F(\tilde{G})} \), if \( |H| \equiv 0 \mod 2 \), then \( \psi(H) = \varphi(H) \). For \( H \in 2^{F(\tilde{G})} \), if \( |H| \equiv 1 \mod 2 \), then \( \psi(H) = V(\tilde{G}) \setminus \varphi(H) = V(\tilde{G}) + \varphi(H) \).

**Assertion 2** The map \( \psi \) is a homomorphism.
Proof. For $H_1, H_2 \in 2^{F(\tilde{G})}$, if $|H_1 + H_2| \equiv 0 \text{ mod } 2$, then by Proposition 2.1, $\psi(H_1 + H_2) = \varphi(H_1 + H_2)$. Since $\varphi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2)$ by Assertion 1 in Subsection 2.4, $\psi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2)$. Then by Remark 2.1, we have the following 2 cases.

Case 1: $|H_1| \equiv 0 \text{ mod } 2$ and $|H_2| \equiv 0 \text{ mod } 2$.

By Proposition 2.1, we have $\varphi(H_1) = \psi(H_1)$ and $\varphi(H_2) = \psi(H_2)$. Then this fact implies that $\psi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2)$.

Case 2: $|H_1| \equiv 1 \text{ mod } 2$ and $|H_2| \equiv 1 \text{ mod } 2$.

By Proposition 2.1, we have $\varphi(H_1) = \psi(H_1) + V(\tilde{G})$ and $\varphi(H_2) = \psi(H_2) + V(\tilde{G})$. Then this fact implies that $\psi(H_1 + H_2) = (\psi(H_1) + V(\tilde{G})) + (\psi(H_2) + V(\tilde{G})) = \psi(H_1) + \psi(H_2)$.

If $|H_1 + H_2| \equiv 1 \text{ mod } 2$, then by Proposition 2.1, $\psi(H_1 + H_2) = \varphi(H_1 + H_2) + V(\tilde{G})$. Since $\varphi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2)$ by Assertion 1, $\psi(H_1 + H_2) = \varphi(H_1) + \varphi(H_2) + V(\tilde{G})$. Then by Remark 2.1, we have the following 2 cases.

Case 3: $|H_1| \equiv 1 \text{ mod } 2$ and $|H_2| \equiv 0 \text{ mod } 2$.

By Proposition 2.1, we have $\varphi(H_1) = \psi(H_1) + V(\tilde{G})$ and $\varphi(H_2) = \psi(H_2)$. Then this fact implies that $\psi(H_1 + H_2) = \psi(H_1) + \psi(H_2)$.

Case 4: $|H_1| \equiv 0 \text{ mod } 2$ and $|H_2| \equiv 1 \text{ mod } 2$.

The argument is essentially the same as that of Case 3, and we omit it.

These complete the proof of Assertion 2. □

Recall that $D$ is a link diagram such that $|D|$ represents $\tilde{G}$. Then we have;

**Proposition 2.2.** Let $H$ be a set of faces of $\tilde{G}$. Then $\psi(H)$ corresponds to the set of the crossings changed by region freeze crossing change at $H$.

**Proof.** Inoue-Shimizu proved the following;

**Claim 3** ([8] Lemma 3.1.) Let $D$ be a link diagram and $H$ a set of regions of $D$. If $|H|$ is even, then effects of region freeze crossing changes at $H$ equals to effects of region crossing changes at $H$. If $|H|$ is odd, then the effect of region freeze crossing changes at $H$ is obtained from “the image of the link diagram obtained from $D$ by applying region crossing changes at $H$” by changing all of the crossings.

Suppose that $|H| \equiv 0 \text{ mod } 2$. Then we have $\psi(H) = \varphi(H)$ by Proposition 2.1. Hence by Claim 2 and Claim 3, we see that $\psi(H)$ corresponds to the set of the crossings changed by the region freeze crossing changes at $H$. 

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Suppose that $|H| \equiv 1 \mod 2$. Then we have $\psi(H) = \varphi(H) + V(\tilde{G})$ by Proposition 2.1. Hence by Claim 2, and Claim 3, we see that $\psi(H)$ corresponds to the set of the crossings changed by the region freeze crossing changes at $H$.

□
3. Structure of the Homomorphism $\varphi$

Let $\tilde{G}, V(\tilde{G}), F(\tilde{G})$ be as in Subsection 2.3. Let $D$ be a link diagram such that $|D|$ represents $\tilde{G}$. Let $\varphi : 2^F(\tilde{G}) \to 2^V(\tilde{G})$ be as in Subsection 2.4. For a vertex $v$ and a set of faces $H$, let $d_v(H)$ be as in Subsection 2.4.

3.1. $\ker \varphi$. In this subsection, we study $\ker \varphi$. We first note that by Claim 2 in Subsection 2.4, we have

Assertion 3. $\ker \varphi$ corresponds to the set of the coloring $H$ such that the region crossing change at $H$ dose not change given link diagram $D$.

We say that a coloring of $\tilde{G}$ is a checker board coloring of $\tilde{G}$ if the following condition is satisfied.

For any point $p$ on $\tilde{G} \setminus V(\tilde{G})$, a small disk neighborhood of $p$ is divided into two parts by $\tilde{G}$. Then one part is shaded and the other is non-shaded as in Figure 23.

Remark 3.1. It is well known that each $\tilde{G}$ admits exactly two checkerboard colorings.

Let $\tilde{G}_1, \ldots, \tilde{G}_n$ be the cut-through circuits of $\tilde{G}$. We fix a point $\infty$ in $S^2 \setminus \tilde{G}$. Then $F(\tilde{G}_i)$ denotes the set of the faces of $\tilde{G}_i$. Further $\xi_i$ denotes the natural map $2^F(\tilde{G}_i) \to 2^F(\tilde{G})$.

Definition 3.1. Let $\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_n$, $F(\tilde{G}_i), F(\tilde{G}), \xi_i$ be as above. A coloring of $\tilde{G}$ is called a componentwise checkerboard coloring associated with $\tilde{G}_i$, if the coloring is the image of a checkerboard coloring of $\tilde{G}_i$ by $\xi_i$. Then $B_i$ denotes the componentwise checkerboard coloring associated with $\tilde{G}_i$, such that the face containing $\infty$ is not shaded.

For the sake of simplicity, we denote $F(\tilde{G})$ by $B_0$.

Remark 3.2. Suppose that $\tilde{G}$ is irreducible. Then, by Observation 2.2, we see that for each $i$, region crossing change at the set of regions corresponding to $B_i$ does not change $D$. 

\begin{figure}[h]
\centering
\includegraphics[width=5cm]{figure23.png}
\caption{Figure 23}
\end{figure}
We prove the following theorem:

**Theorem 1.2’.** [5]

Let \( \tilde{G} \) be a union of mutually disjoint 4-valent plane graph(s) or circle(s). Let \( V(\tilde{G}), F(\tilde{G}), \varphi : 2^F(\tilde{G}) \to 2^V(\tilde{G}), \tilde{G}_i \ (1 \leq i \leq n), B_i \ (0 \leq i \leq n) \) be as above. Suppose that \( \tilde{G} \) is irreducible. Then \( \{B_0, B_1, \ldots, B_n\} \) is a basis of \( \ker \varphi \).

**Lemma 3.1.** Suppose that \( \tilde{G} \) is irreducible. Then \( \{B_0, B_1, \ldots, B_n\} \) is linearly independent.

**Proof.** Suppose that the following equation is satisfied.

\[
\varepsilon_0 B_0 + \varepsilon_1 B_1 + \varepsilon_2 B_2 + \cdots + \varepsilon_n B_n = 0 \quad (\varepsilon_i \in \mathbb{Z}_2)
\]

Recall that a point \( \infty \) in \( S^2 \setminus \tilde{G} \) is fixed, and the face containing \( \infty \) is not shaded by \( B_i \) \((1 \leq i \leq n)\). On the other hand, the region containing \( \infty \) is shaded by \( B_0 \). These together with (4) implies \( \varepsilon_0 = 0 \). Then, for each \( i \ (1 \leq i \leq n) \), we take a point \( p_i \) in \( \tilde{G} \setminus V(\tilde{G}) \) such that \( p_i \in \tilde{G}_i \). A small disk neighborhood of \( p_i \) is divided into two components by \( \tilde{G} \). One component is contained in a face shaded by \( B_i \), say \( R_s \), and the other is not contained in any face shaded by \( B_i \). Then \( R_t \) denotes the face containing the other component (Figure 24(a)). On the other hand, we see that both of the faces \( R_s \) and \( R_t \) are shaded or non-shaded by \( B_j \ (j \neq i, j \in \{1, \ldots, n\}) \) by the definition of \( B_j \) (Figure 24(b)). These facts together with (4) imply \( \varepsilon_i = 0 \).

![Figure 24](image-url)

As the conclusion, we have shown that \( \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_n = 0 \). Hence \( \{B_0, B_1, \ldots, B_n\} \) is linearly independent. \( \square \)

Now we give a proof of Theorem 1.2’. Suppose that \( \tilde{G} \) is irreducible. Then by Remark 3.2 and Assertion 3, \( B_i \in \ker \varphi \ (i = 0, 1, \ldots, n) \). By Lemma 3.1, \( \{B_0, B_1, \ldots, B_n\} \) is linearly independent. This fact together with (3) in Section 2.4 implies \( \{B_0, B_1, \ldots, B_n\} \) is a basis of \( \ker \varphi \). This completes the proof of Theorem 1.2’.
3.2. $\text{Im}\varphi$. In this subsection, we give a family of set of vertices of $\tilde{G}$ that generates $\text{Im}\varphi$. We start with technical results.

It is not philosophically difficult to generalize Theorem 1.2' for reducible $\tilde{G}$, but it is not easy to give a concrete basis with the notations something like $\{B_0, B_1, \ldots, B_n\}$. However this type of generalization is required for giving the generating family and the next proposition gives a technically least property required for the proof. In general, for any $p \in \tilde{G} \setminus V(\tilde{G})$, there are 4 patterns of local coloring of (a small disk neighborhood of) $p$ as in Figure 25.

**Figure 25**

**Proposition 3.1.** Let $\tilde{G}$ be a union of mutually disjoint 4-valent plane graph(s) or circle(s). For any point $p \in \tilde{G} \setminus V(\tilde{G})$, we have: For any local coloring of a small disk neighborhood of $p$, say $N(p)$, there is a coloring $H$ of $\tilde{G}$ such that

1. $H \in \ker \varphi$
2. the restriction of $H$ to $N(p)$ coincides with the given one.

**Proof.** The proof is by the induction on the number of the connected components of $\tilde{G}$, denoted by $k$.

Suppose that $k = 1$. In this case, we prove the proposition by the induction on the number of the reducible vertices of $\tilde{G}$, denoted by $m$. First suppose that $m = 0$. Note that $N(p)$ is divided into two components by $\tilde{G}$. Then by Theorem 1.2', we have the following: If the local coloring of $p$ is as Figure 26 (a), then $\emptyset \in 2^{F(\tilde{G})}$ satisfies the conditions of Proposition 3.1. If the local coloring of $p$ is as Figure 26 (b), then $F(\tilde{G}) \in 2^{F(\tilde{G})}$ satisfies the conditions. Suppose that the local coloring of $p$ is as Figure 26 (c). Let $H$ be the componentwise checkerboard coloring of $\tilde{G}$ associated with the cut-through circuit of $\tilde{G}$ which contains $p$. Then, by Theorem 1.2', we see that one of $H \in 2^{F(\tilde{G})}$ or $H + F(\tilde{G}) \in 2^{F(\tilde{G})}$ satisfies the conditions.

Next we suppose that $m > 0$. Let $G'$ be the graph obtained from $\tilde{G}$ by smoothing a reducible vertex denoted by $c$. Since $c$ is a reducible vertex, $G'$ consists of two connected components, say $G^{(1)}$ and $G^{(2)}$ where $p \in G^{(1)}$. Let $E$ be a circle on $S^2$ separating $G^{(1)}$ and $G^{(2)}$. Note that $N(p)$ is regarded as a neighborhood of $p \in G^{(1)}$ and we may suppose that $N(p) \cap G^{(1)} = N(p) \cap \tilde{G}$. By the assumption of the induction there exists a coloring $H^{(1)}$ on $G^{(1)}$ respecting the local coloring on $N(p)$ such that $H^{(1)} \in \ker \varphi^{(1)}$, where $\varphi^{(1)}$: 23
2^{F(G^{(1)})} \to 2^{V(G^{(1)})} is the homomorphism defined as in Section 2. Let \( N(c) \) be a small disk neighborhood of \( c \) such that \( N(c) \cap G' \) is formed by two subarcs \( \alpha_1, \alpha_2 \) properly embedded in \( N(c) \) such that \( \alpha_1 \subset G^{(1)}, \alpha_2 \subset G^{(2)} \). Further, fix a point \( p_i \in \text{Int} \alpha_i \) for \( i = 1, 2 \). We may regard \( p_i \in G^{(i)} \setminus V(G^{(i)}) \), and \( N(p_i) = N(c) \). Then the coloring \( H^{(1)} \cap N(p_1) \) is one of the 4 patterns in Figure 27. According to each pattern of \( H^{(1)} \cap N(p_1) \), we take the local coloring at \( p_2 \) as in Figure 27.

By the assumption of the induction, for each local coloring at \( p_2 \), there exists a coloring \( H^{(2)} \) on \( G^{(2)} \) respecting the local coloring such that \( H^{(2)} \in \ker \varphi^{(2)} \), where \( \varphi^{(2)} : 2^{F(G^{(2)})} \to 2^{V(G^{(2)})} \) is the homomorphism defined as in Section 2.4. Then it is directly observed from Figure 28 that we can amalgamate the colorings \( H^{(1)} \) and \( H^{(2)} \) to give a coloring of \( \tilde{G} \).

By this construction and Observation 2.2, we see that this coloring of \( \tilde{G} \) satisfies the conditions of Proposition 3.1. This completes the proof of the proposition for the case of \( k = 1 \).

Next suppose that \( k > 1 \). Let \( E' \) be a circle on \( S^2 \) such that \( E' \cap \tilde{G} = \emptyset \), and that \( E' \) separates \( \tilde{G} \) into two non-empty subsets, denoted by \( \tilde{G}_1^c, \tilde{G}_2^c \), where \( p \in \tilde{G}_1^c \). By the assumption of the induction, for each local coloring at \( p \), there exists a coloring
$H_1 \in 2^{F(\tilde{G}_i)}$ such that $H_1$ satisfies the conditions of Proposition 3.1, i.e., $H_1 \in \ker \varphi_1$ and the restriction of the coloring of $H_1$ to $N(p)$ coincides with the given one, where $\varphi_1 : 2^{F(\tilde{G}_i)} \to 2^{V(\tilde{G}_i)}$ is the homomorphism defined as in Section 2.4. Let $F_i^c$ be the face of $\tilde{G}_i^c$ such that $E' \subset F_i^c$ $(i = 1, 2)$. Fix a point $p_i$ contained in $\tilde{G}_i^c \setminus V(\tilde{G}_i^c)$ on the boundary $F_i^c$. Then the coloring $H_1 \cap N(p_1)$ is one of the 4 patterns in Figure 28. According to each pattern of $H_1 \cap N(p_1)$, we take the local coloring at $p_2$ as in Figure 28 in the proof. However, we may take the local coloring of $p_2$ as in Figure 29. The point is that the color of $F_1^c$ and the color of $F_2^c$ coincide.

Recall that $\tilde{G}_i$ $(1 \leq i \leq n)$ is a cut-through circuit of $\tilde{G}$. We note that for any subset of the cut-through circuits, say $\{\tilde{G}_{i_1}, \ldots, \tilde{G}_{i_k}\}$, $\tilde{G}_{i_1} \cup \cdots \cup \tilde{G}_{i_k}$ is a union of mutually disjoint 4-valent plane graph(s) or circle(s), hence the notation $V(\tilde{G}_{i_1} \cup \cdots \cup \tilde{G}_{i_k})$ makes sense. We will use the following notation; $V_{ij} = V(\tilde{G}_i \cup \tilde{G}_j) \setminus (V(\tilde{G}_i) \cup V(\tilde{G}_j))$ $(i \neq j)$. See Figure 30. (We note that $V_{ij}$ corresponds to the set $\mathcal{C}_{ij}$ in Section 1.) Here, we define a terminology corresponding to $\mathcal{T}^s$ in Section 1. We abuse notation to use the same symbol $\mathcal{T}^s$ to represent it.

$\square$

**Remark 3.3.** According to each local coloring of $p_1$, we took the local coloring $p_2$ as in Figure 28 in the proof. However, we may take the local coloring of $p_2$ as in Figure 29. The point is that the color of $F_1^c$ and the color of $F_2^c$ coincide.
Definition 3.2. Let $\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_n$ be as above and $s \in \{2, \ldots, n\}$. A set of mutually different $s$ vertices $\{c_1, \ldots, c_s\} (\subset V(\tilde{G}))$ is a $v$-cycle of length $s$, if there exists $\{p_1, p_2, \ldots, p_s\} \subset \{1, 2, \ldots, n\}$ ($p_t \neq p_u$, for each $t \neq u$) such that $c_1 \in V_{p_1p_2}$, $c_2 \in V_{p_3p_4}$, $\cdots$, $c_{s-1} \in V_{p_{s-1}p_s}$, $c_s \in V_{p_sp_1}$. Then $\mathcal{T}^s (\subset 2^V(\tilde{G}))$ denotes the set consisting of the $v$-cycles of length $s$. Further, for each $c \in V(\tilde{G}_i)$ ($i = 1, \ldots, n$), $\{c\}$ is called a $v$-cycle of length 1, and $\mathcal{T}^1$ denotes the set of the $v$-cycles of length 1.

With these notations, Propositions 2.2 of [3] can be phrased as follows.

**Proposition 3.2.** Let $\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_n$ be as above. Then for each $c \in V(\tilde{G}_i)$, we have $\{c\} \in \text{Im}\varphi$, which implies $\mathcal{T}^1 \subset \text{Im}\varphi$.

**Proposition 3.3.** Let $\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_n$ be as above. Then for each pair of vertices $c_1, c_2 \in V_{ij} (c_1 \neq c_2)$, we have $\{c_1, c_2\} \in \text{Im}\varphi$, which implies $\mathcal{T}^2 \subset \text{Im}\varphi$.

In [2] (Proposition 4.2), Propositions 3.2, 3.3 are generalized as follows.
Proposition 3.4. Let \( \tilde{G} = G_1 \cup \cdots \cup G_n \) be as above. Then for each \( s \) \((1 \leq s \leq n)\), \( T^s \subset \text{Im} \varphi \).

In Subsection 3.4, we give an alternative proof of Proposition 3.4 in the proof of Proposition 3.5. Note that Proposition 3.4 implies that \( \mathcal{H} \subset \text{Im} \varphi \), where \( \mathcal{H} \) denotes the subspace of \( 2^{V(\tilde{G})} \) generated by \( T^1 \cup T^2 \cup \cdots \cup T^n \). Clearly, the next theorem is equivalent to Theorem 1.3 in Section 1. In Subsection 3.4, we prove Theorem 1.3′ for Theorem 1.3.

Theorem 1.3′[6] Let \( \tilde{G}, \varphi, \mathcal{H} \) be as above. Then
\[
\mathcal{H} = \text{Im} \varphi.
\]

3.3. Coker \( \varphi \). In this subsection, we study Coker \( \varphi \). Let \( \tilde{G}, V(\tilde{G}), F(\tilde{G}), \varphi : 2^F(\tilde{G}) \to 2^{V(\tilde{G})} \) and \( \tilde{G}_i \) \((1 \leq i \leq n)\) be as above.

Here note that in our setting, Coker \( \varphi \) coincides with the coset decomposition of \( 2^{V(\tilde{G})} \) by \( \text{Im} \varphi \) as an abelian group. With this fact in mind, it is easy to see that if \( \tilde{G} = \tilde{G}_1 \sqcup \tilde{G}_2 \), then the coset decomposition of \( 2^{V(\tilde{G})} \) by \( \text{Im} \varphi \) is naturally induced by the coset decompositions of \( 2^{V(\tilde{G}_i)} \) by \( \text{Im} \varphi_i \) \((i = 1, 2)\), where \( \varphi_i : F(\tilde{G}_i) \to V(\tilde{G}_i) \) is the homomorphism as in Subsection 2.4. Hence it is enough to assume that \( \tilde{G} \) is connected.

Let \( G = G_1 \cup \cdots \cup G_n \) be a connected 4-valent plane graph, where \( G_1, \ldots, G_n \) are the cut-through circuits of \( G \). For the statement of our result, we introduce a graph and some notations derived from the 4-valent plane graph \( G \). Let \( \mathcal{G}_G \) be the graph obtained from \( G \) as follows.

The set of the vertices \( V(\mathcal{G}_G) \) consists of \( n \) elements \( v_1, \ldots, v_n \) such that \( v_i \)
corresponds to the cut-through circuit \( G_i \) of \( G \).

There is an edge joining \( v_i \) and \( v_j \) if and only if \( V_{ij} (= V(G_i \cup G_j) \setminus (V(G_i) \cup V(G_j))) \neq \emptyset \).

Note that since \( G \) is connected, \( \mathcal{G}_G \) is connected. Then, let \( T_G \) be a spanning tree of \( \mathcal{G}_G \), that is, \( T_G \) is a subtree of \( \mathcal{G}_G \) such that \( V(T_G) = V(\mathcal{G}_G) = \{v_1, \ldots, v_n\} \). Let \( \{e_1, \ldots, e_{n-1}\} \) be the set of edges of \( T_G \). Fix a vertex \( d_i \) of \( G \) corresponding to \( e_i \) \((i = 1, \ldots, n - 1)\). Clearly, the next theorem is equivalent to Theorem 1.4 in Section 1. In Subsection 3.4, we prove Theorem 1.4′ for Theorem 1.4.

Theorem 1.4′[6] Let \( G, \varphi, d_i \) \((i = 1, \ldots, n - 1)\) be as above. We regard \( 2^{V(G)} \) as a group whose operation is given by symmetric difference. Then, the coset decomposition of \( 2^{V(G)} \) by \( \text{Im} \varphi \) has the following presentation:
\[
2^{V(G)} = \prod_{Y \in 2^{\{d_1, \ldots, d_{n-1}\}}} (Y + \text{Im} \varphi).
\]
Example 3.1. We consider the plane graph $G$ which can be regarded as a projection of a Borromean rings in Figure 31 (a). By Theorem 1.4 in [3], we see that $\dim(\text{Im}\varphi) = 4$. This shows that $\text{Im}\varphi$ consists of 16 elements. By using Theorem 1.3, we can show that they are represented by the 16 sets of vertices which are encircled by upper-left square in Figure 32. Each figure in Figure 32 express the set of vertices represented by dots. Next we discuss about $\text{Coker}\varphi$. Note that the graph $G_G$ of $G$ is as Figure 31 (b). Take the spanning tree $T_G$ as in Figure 31 (c), where $E(T_G) = \{e_1, e_2\}$. Then fix a vertex $d_i$ ($i = 1, 2$) of $G$ corresponding to $e_i$ as in Figure 31 (d). Since $G$ has 6 vertices, $|2^V(G)| = 2^6$. Figure 32 gives all of the elements of $2^V(G)$. Among them, the 16 sets at upper-left square in Figure 32 gives $\text{Im}\varphi$ as remarked above. Then Theorem 1.4’ says that the coset decomposition of $2^V(G)$ by $\text{Im}\varphi$ are given by subset of $\{d_1, d_2\}$, i.e. we see that $2^V(G) = \text{Im}\varphi \amalg (\{d_1\} + \text{Im}\varphi) \amalg (\{d_2\} + \text{Im}\varphi) \amalg (\{d_1, d_2\} + \text{Im}\varphi)$ which is represented by Figure 32.

Figure 31

(a) (b) (c) (d)
3.4. **Proofs of Theorems 1.3 and 1.4.** In this subsection, we give proofs of Theorems 1.3’ and 1.4’, and these immediately give proofs of Theorems 1.3 and 1.4. We first note that for the proof of Theorem 1.3’, it is enough to suppose that $\tilde{G}$ is connected. (We have already remarked that we may suppose that $\tilde{G}$ is connected for the proof of Theorem 1.4’.) In fact, if $\tilde{G} = \tilde{G}_1 \amalg \tilde{G}_2$, then it is easy to see that $\text{Im}\varphi$ is naturally identified with $\text{Im}\varphi_1 \times \text{Im}\varphi_2$ and $\mathcal{H}$ is naturally identified with $\mathcal{H}_1 \times \mathcal{H}_2$. These show that it is enough to prove Theorem 3 for connected $\tilde{G}$. Let $G = G_1 \cup \cdots \cup G_n$ be a connected 4-valent graph consisting of $n$ cut-through circuits.

Now we first prove;

**Proposition 3.5.** Let $\varphi, \mathcal{H}$ be as in Section 2. Then,

$$\mathcal{H} \subset \text{Im}\varphi.$$ 

*Proof.* First, recall that any $v$-cycle of length 1 is an element of $\text{Im}\varphi$ (Proposition 3.2).
Hence we consider \( \nu \)-cycles of length \( s \) for each \( s \geq 2 \). For the proof of Proposition 3.5, it is enough to show that for \( \{c_1, \ldots, c_s\} \in T^* \), we have \( \{c_1, \ldots, c_s\} \in \text{Im}\varphi \). We note that this assertion was stated in [2] and an outline of a proof is given there (Proposition 4.2). Here we will give a proof of this by using different arguments. Let \( V_{ij} \) be as in Section 2. By exchanging subscripts if necessary, we may suppose that \( c_1 \in V_{12}, c_2 \in V_{23}, \ldots, c_{s-1} \in V_{s-1s}, c_s \in V_{s1} \). Let \( m \) be the number of reducible vertices of \( G \). We prove this proposition by the induction on \( m \).

First suppose that \( m = 0 \). By smoothing \( c_1, \ldots, c_{s-1} \) in this order, \( G_1 \cup \cdots \cup G_s \) becomes a 4-valent plane graph consisting of one cut-through circuit or a circle (because by smoothing \( c_1 \), \( G_1 \) and \( G_2 \) are amalgamated, and by repeating smoothings \( c_2, \ldots, c_{s-1} \) in order, \( G_1, \ldots, G_s \) are amalgamated to a 4-valent plane graph consisting of one cut-through circuit or a circle.) Finally, by smoothing \( c_s \), we obtain union of 4-valent plane graph(s) or circle(s) say \( G' \cup G'' \), such that \( (G' \cup G'') \cap N(c_s) \) is formed by two subarcs, one of which is included in \( G' \) and the other is included in \( G'' \) (see Figure 33).

Let \( G^* = G' \cup G'' \cup G_{s+1} \cup \cdots \cup G_n \). Let \( H^* (\in 2^{F(G^*)}) \) be a componentwise checkerboard coloring associated with \( G' \). A local coloring of \( c_s \) induced by \( H^* \) is as in Figure 34 (because \( N(c_s) \cap G^* \) is formed by two subarcs one of which is included in \( G' \) and the other is included in \( G'' \)). Since the consequences of the smoothings at \( c_1, \ldots, c_s \) are independent from the orders of the smoothings, by taking \( c_i \ (i = 1, \ldots, s-1) \) for \( c_s \) in the above argument, we see that for each \( i \), \( H^* \) looks as one of the figures in Figure 34 in a neighborhood of \( c_i \).

Let \( c' \in V(G^*) \). Then \( N(c') \cap G^* \) is formed by two subarcs, say \( \beta_1, \beta_2 \). If both \( \beta_1 \) and \( \beta_2 \) are not contained in \( G' \), then the local coloring of \( c' \) is as one of the figures in Figure 35.
If either one of $\beta_1$ or $\beta_2$ is contained in $G'$, then the local coloring of $c'$ is as one of the figures in Figure 36.

![Figure 35](image1)

![Figure 36](image2)

Then let $H$ be the coloring of $G$ introduced from $H^*$ (see Figure 37). Since $G$ is irreducible, these show, by Observation 2.2, that $\varphi(H) = \{c_1, \ldots, c_s\}$, that is \(\{c_1, \ldots, c_s\} \in \text{Im} \varphi\).

Next, we suppose that $m \geq 1$. Let $c$ be a reducible vertex of $G$. Note that by smoothing $G$ at $c$, we obtain a disconnected graph $G^{(1)} \amalg G^{(2)}$, where $G^{(1)}, G^{(2)}$ are connected components. Since each $c_i$ is formed by two subarcs from two different cut-through circuits of $G$, we see that $c_i$ is not a reducible vertex. Hence $c_1, \ldots, c_s \in V(G^{(1)} \amalg G^{(2)})(= V(G^{(1)}) \amalg V(G^{(2)}))$.

**Claim 4.** All of $c_1, \ldots, c_s$ are contained in $G^{(1)}$ or $G^{(2)}$.

**Proof.** Suppose that the claim does not hold. By changing indices, if necessary, we may suppose that $c_1 \in G^{(1)}$ and $c_2 \in G^{(2)}$. Since $c_1 \in G_1 \cap G_2 (\subset G_2) \cap G_3 (\subset G_2)$, we see that $N(c_2) \cap \hat{G}$ is formed by two subarcs of $G_2$. Since $c_2 \in G_2 \cap G_3$, we see that $G_3 \subset G^{(2)}$. Since $c_3 \in G_3 \cap G_4$, we see that $G_4 \subset G^{(2)}$. By repeating the same argument, we see that $G_s \subset G^{(2)}$. Further, since $c_s \in G_s \cap G_1$, we see that $G_1 \subset G^{(2)}$ hence, $c_1 \in G^{(2)}$, a contradiction. \qed

Without loss of generality, we may suppose that $G^{(1)}$ contains $c_1, \ldots, c_s$. Since the number of the reducible vertices of $G^{(1)}$ is less than $m$, by the assumption of the induction, there exists a coloring $H^{(1)}$ of $G^{(1)}$ such that $\varphi^{(1)}(H^{(1)}) = \{c_1, \ldots, c_s\}$ where $\varphi^{(1)} : 2^{E(G^{(1)})} \rightarrow 2^{V(G^{(1)})}$ is the homomorphism defined as in Section 2.4. Let $N(c)$ be a small disk neighborhood of $c$ such that $N(c) \cap G^*$ is formed by two subarcs $\alpha^{(1)}, \alpha^{(2)}$.
such that $\alpha^{(1)} \subset G^{(1)}$, $\alpha^{(2)} \subset G^{(2)}$. Further, fix a point $p_i \in \text{Int}^{(i)}$ ($i = 1, 2$). We may regard $N(p_i) = N(c)$. Then in $G^{(1)}$, the local coloring of $N(c)$ induced by $H^{(1)}$ is one of 4 patterns in Figure 38. According to each pattern, take the local coloring in Figure 39 with each figure corresponding to that in Figure 38 with the same label. By Proposition 3.1, according to each pattern, there is a coloring $H^{(2)}$ in $G^{(2)}$ such that $H^{(2)} \in \ker \varphi^{(2)}$, where $\varphi^{(2)} : 2^{F(G^{(2)})} \to 2^{V(G^{(2)})}$ is the homomorphism defined as in Section 2. Then it is directly observed that from Figure 38 and Figure 39, we can amalgamate the colorings $H^{(1)}$ and $H^{(2)}$ to give a coloring $H$ as in Figure 17. By this construction and Observation 2.1, we see that the coloring $H$ satisfies $\varphi(H) = \{c_1, \ldots, c_s\}$, that is, $\{c_1, \ldots, c_s\} \in \text{Im} \varphi$. This completes the proof of the proposition. \qed

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
\includegraphics[width=0.2\textwidth]{fig38a} & \includegraphics[width=0.2\textwidth]{fig38b} & \includegraphics[width=0.2\textwidth]{fig38c} & \includegraphics[width=0.2\textwidth]{fig38d} \\
(1) & (2) & (3) & (4)
\end{tabular}
\caption{Figure 38}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
\includegraphics[width=0.2\textwidth]{fig39a} & \includegraphics[width=0.2\textwidth]{fig39b} & \includegraphics[width=0.2\textwidth]{fig39c} & \includegraphics[width=0.2\textwidth]{fig39d} \\
(1) & (2) & (3) & (4)
\end{tabular}
\caption{Figure 39}
\end{figure}

Now we prove Theorems 1.3’ and 1.4’. Let $V(G), F(G), \varphi : 2^{F(G)} \to 2^{V(G)}; \mathcal{H}, \mathcal{G}_G, d_1, d_2, \ldots, d_{n-1}$ be as in Section 3.3. The degree of a vertex $v$ of $G$ is denoted by $\deg_G(v)$. For any set $J$ of vertices of $G$, let $S_J$ be the subgraph of $\mathcal{G}_G$ obtained from $J$ as follows; $V(S_J) = \{v_1, \ldots, v_n\} = V(G)$, i.e. each vertex $v_i$ of $S_J$ corresponds to the cut-through circuit $G_i$ of $G$ and for a pair of vertices $\{v_i, v_j\}$ ($i, j = 1, \ldots, n, i \neq j$), there is an edge between $v_i$ and $v_j$ if and only if $|J \cap V_{ij}|$ is an odd integer. (Note that there may be an isolated vertex in $S_J$.)
Remark 3.4. By the definition of $v$-cycles and the construction of $S_J$, we see that:

1. If $J$ is a $v$-cycle of length 1 or 2, then $E(S_J) = \emptyset$.
2. If $J$ is a $v$-cycle of length $s$ ($s \geq 3$), then $S_J$ consists of a cycle of length $s$, and $n - s$ isolated vertices.

We define a map $\omega : 2^{V(G)} \to \mathbb{Z}_2^{n-1}$ as follows: For $J \in 2^{V(G)}$, $\omega(J) = (\epsilon_1(J), \ldots, \epsilon_{n-1}(J))$ with $\epsilon_i(J) = \eta(\sum_{j=1, \ldots, n, j \neq i} |J \cap V_i|)$ where $\eta : \mathbb{Z} \to \mathbb{Z}_2$ is the epimorphism. It is easy to see that $\omega$ is a $\mathbb{Z}_2$-linear map.

Claim 5. For each $J \subset V(G)$, we have:

$$\epsilon_i(J) = \eta(\deg_{S_J}(v_i)) \quad (i = 1, \ldots, n - 1).$$

Proof. For the proof of Claim 5, it is enough to show that

$$\deg_{S_J}(v_i) \text{ odd } \iff \epsilon_i(J) = 1.$$

First, we suppose that $r := \deg_{S_J}(v_i)$ is an odd number. Let $v_{t_1}, \ldots, v_{t_r}$ be the vertices in $S_J$ which are adjacent to $v_i$, hence $|J \cap V_{t_l}|$ is odd, for each $l \in \{1, \ldots, r\}$ and if $t \not\in \{i, t_1, \ldots, t_r\}$, then $|J \cap V_t|$ is even. By these facts, we have $\sum_{j=1, \ldots, n, j \neq i} |J \cap V_i|$ odd, which shows that $\epsilon_i(J) = 1$.

Next we suppose that $\epsilon_i(J) = 1$, which means that $\sum_{j=1, \ldots, n, j \neq i} |J \cap V_i|$ is odd. Let $\{t_1, \ldots, t_r\}$ be the subset consisting of $j \in \{1, \ldots, n\} \setminus \{i\}$ such that $|J \cap V_j|$ is odd. Since $\sum_{j \neq i} |J \cap V_j|$ is odd, we see that $r$ is an odd number. Here we note that $\deg_{S_J}(v_i) = r$ and these show that $\deg_{S_J}(v_i)$ is odd. $\square$

Then we show;

Claim 6. The map $\omega$ is an epimorphism.

Proof. Since $G$ is connected, for each $v_i \in V(T_G) \setminus \{v_n\}$, there exists a path $P \subset T_G$ joining $v_i$ and $v_n$. Let $J \subset \{d_1, \ldots, d_{n-1}\}$ be the set of the vertices corresponding to the edges of $P$. Note that $S_J$ is represented by $P$ and some isolated vertices. Then, by using Claim 5, we see that $\epsilon_i(J) = \eta(\deg_{S_J}(v_i)) = 1$. Further, for $j \neq i, n$, $\deg_{S_J}(v_j) = 0$ or 2, hence $\epsilon_j(J) = 0$. Therefore for each $i$ ($i = 1, \ldots, n - 1$), we can take a set $J$ such that $\omega(J) = (\epsilon_1(J), \ldots, \epsilon_{n-1}(J))$ with $\epsilon_i(J) = 1, \epsilon_j(J) = 0$ ($j \neq i$). This shows that the image of $\omega$ contains canonical basis of $\mathbb{Z}_2^{n-1}$, and this shows that $\omega$ is an epimorphism. $\square$

For each $(\epsilon_1, \ldots, \epsilon_{n-1}) \in \mathbb{Z}_2^{n-1}$, let $J_{\epsilon_1, \ldots, \epsilon_{n-1}} = \omega^{-1}(\epsilon_1, \ldots, \epsilon_{n-1})$. Then, we immediately have;

$$2^{V(G)} = \bigsqcup_{(\epsilon_1, \ldots, \epsilon_{n-1}) \in \mathbb{Z}_2^{n-1}} J_{\epsilon_1, \ldots, \epsilon_{n-1}}$$

(5)

Further we have;
Lemma 3.2. Let $\mathcal{H}$ be as in Section 3.2. Then $\mathcal{H} = J_{0,0,\ldots,0}$.

Proof. First we show that $\mathcal{H} \subset J_{0,0,\ldots,0}$. Since $\mathcal{H}$ is generated by $v$-cycles of length $s$ ($s = 1, \ldots, n$), it is enough to show, if $J$ is a $v$-cycle of length $s$, we have $J \in J_{0,0,\ldots,0}$. By Remark 3.4 (1), if $s = 1, 2$, then $S_J$ does not contain an edge. By Remark 3.4 (2), if $s \geq 3$, then $S_J$ is represented as a cycle and isolated vertices. In either case the degree of each vertex of $S_J$ is either 0 or 2. Hence by using Claim 2, we have $\omega(J) = (0, 0, \ldots, 0)$. Hence $J \in J_{0,0,\ldots,0}$.

Now we show that $J_{0,0,\ldots,0} \subset \mathcal{H}$. Let $J \in J_{0,0,\ldots,0}$. Recall that $G_i (i = 1, \ldots, n)$ is a cut-through circuit of $\mathcal{H}$. Suppose that $c \in V(G_i)$. Since $\{c\} \in \mathcal{H}$, $J \in \mathcal{H}$ if and only if $J + \{c\} \in \mathcal{H}$. This fact shows that we may suppose that $J$ does not contain a vertex in $V(G_i) (i = 1, \ldots, n)$. Furthermore, suppose $J$ contains $\{c_{ij}, c_{ji}\}$ with $c_{ij}, c_{ji} \in V_{ij}, c_{ij} \neq c_{ji}$. Since $\{c_{ij}, c_{ji}\} \in \mathcal{H}$, $J \in \mathcal{H}$ if and only if $J + \{c_{ij}, c_{ji}\} \in \mathcal{H}$, hence we may ignore the pair $\{c_{ij}, c_{ji}\} \subset J$. Hence, we may suppose that $J$ has at most one element of $V_{ij}$ for each pair $i, j(i \neq j)$. Then we show;

Claim 7. $\deg_{S_J}(v_n)$ is an even number.

Proof. For each element $c$ of $J$, $N(c) \cap G$ is formed by two subarcs, say $\alpha_c, \beta_c$. Let $\mathcal{A} = \cup_{c \in J}\{\alpha_c, \beta_c\}$. Since $|J \cap V_{ij}| \leq 1$, there is an edge in $E(S_J)$ between $v_i$ and $v_j$ if and only if $|J \cap V_{ij}| = 1$. This shows that $\deg_{S_J}(v_i)$ is the number of the elements of $\mathcal{A}$ which are subarcs of $G_i$. Hence $|\mathcal{A}| = \deg_{S_J}(v_1) + \deg_{S_J}(v_2) + \cdots + \deg_{S_J}(v_n)$. Then we have:

$$\deg_{S_J}(v_n) = |\mathcal{A}| - \sum_{i=1,\ldots,n-1} \deg_{S_J}(v_i).$$

Since each vertex in $J$ is formed by two subarcs, $|\mathcal{A}| (= 2|J|)$ is an even number. Since $J \in J_{0,0,\ldots,0}$, $\deg_{S_J}(v_i)$ $(i = 1, \ldots, n - 1)$ is an even number by Claim 2. These imply that $\deg_{S_J}(v_n)$ is an even number.

We have established the fact: the degree of each vertex of $S_J$ is even. It is elementary to show that this fact implies that $S_J$ is decomposed into embedded cycles (Figure 40). Let $E$ be such a cycle. Let $v_{i_1}, \ldots, v_{i_s}$ be the vertices contained in $E$ in this order. Let $J_E$ be the subset of $J$ consisting of the elements corresponding to the edges of $E$. Then $J_E$ is a cycle of length $s$, consisting of vertices in $V_{i_1i_2}, V_{i_2i_3}, \ldots, V_{i_{s-1}}$. Hence $J_E \in \mathcal{H}$. Since this assertion holds for each embedded cycle, we have $J \in \mathcal{H}$. This completes the proof of Lemma 3.2.
Then, we see:

**Lemma 3.3.** For any \( J \in 2^{V(G)} \), there exists \( Y \in 2^{\{d_1, \ldots, d_{n-1}\}} \) such that \( J \subseteq Y + J_{0,0,\ldots,0} \).

**Proof.** Recall that \( G_G \) is the graph induced from \( G \) and \( T_G \) the spanning tree of \( G_G \), where \( \{e_1, \ldots, e_{n-1}\} \) is the set of the edges of \( T_G \) with \( e_i \) corresponding to the vertex \( d_i \) \((i = 1, \ldots, n-1)\).

**Claim 8.** There is a subset \( J' \) of \( 2^{V(G)} \) satisfying the following conditions.

\[
\begin{align*}
(a) & \quad J' + J \in J_{0,0,\ldots,0} \\
(b) & \quad |J' \cap V(G_i)| = 0 \quad (i = 1, \ldots, n) \\
(c) & \quad |J' \cap V_{ij}| \leq 1 \quad (i \neq j)
\end{align*}
\]

**Proof.** By the arguments in the paragraph preceding Claim 7 in the proof of Lemma 3.2, we see that there exists a subset \( K \subset J \) such that \( (a) \quad K \in J_{0,0,\ldots,0} \), \( (b) \quad |(J+K) \cap V(G_i)| = 0 \quad (i = 1, \ldots, n) \), and \( (c) \quad |(J + K) \cap V_{ij}| \leq 1 \) \((i \neq j)\). Then \( J' := J + K \) satisfies the condition of Claim 8. \( \square \)

Then \( e_j \ (n-1 < j \leq N) \), where \( N = |E(G_G)| \) denotes the edges of \( G_G \) that are not contained \( E(T_G) \). Then, fix a vertex \( d_j \) of \( G \) corresponding to \( e_j \) \((n-1 < j \leq N)\).

**Claim 9.** There is \( J'' \in 2^{\{d_1, \ldots, d_{n-1}, d_{n}, \ldots, d_N\}} \) such that \( J'' + J' \in J_{0,0,\ldots,0} \).

**Proof.** For each \( v \in J' \), by the property \( (b) \) in Claim 8, \( v \) is a vertex formed by two different cut-through circuits. Further \( \{d_1, \ldots, d_N\} \) corresponds to \( E(G_G) \) bijectively. Hence, for each \( h \in J' \), there exists a unique element \( d_{a(h)} \) in \( \{d_1, \ldots, d_N\} \) such that \( h \) and \( d_{a(h)} \) are contained in the same \( V_{ij} \). We define \( \rho : J' \rightarrow \{d_1, \ldots, d_N\} \) by \( \rho(h) = d_{a(h)} \).
Then let \( J'' = \rho(J') \). Let \( J'_1 = \{ h \in J' \mid d_{s(h)} \neq h \} \) and \( J'_2 = \{ h \in J' \mid d_{s(h)} = h \} \). It is clear that \( J' = J'_1 \sqcup J'_2 = J'_1 + J'_2 \). Since \( \rho : J' \to \{d_1, \ldots, d_N\} \) is injective by (c) in Claim 8, \( J'' = \rho(J') = \rho(J'_1) \sqcup \rho(J'_2) = \rho(J'_1) + \rho(J'_2) \). Further by the definition of \( J'_1 \), we have \( J'_1 \cap \rho(J'_1) = \emptyset \) and by the definition of \( J'_2 \), we have \( J'_2 = \rho(J'_2) \). As a conclusion, \( J'' + J' = \rho(J'_1 + J'_2) + J'_1 + J'_2 = \rho(J'_1) \sqcup J'_1 \). Let \( \{h_1, \ldots, h_k\} = J'_1 \). Then \( J'' = \rho(J'_1) = \{d_{s(h_1)}, \ldots, d_{s(h_k)}\} \). Hence
\[
J'' + J' = \rho(J'_1) \sqcup J'_1
= \{d_{s(h_1)}, \ldots, d_{s(h_k)}\} \sqcup \{h_1, \ldots, h_k\}
= \{h_1, d_{s(h_1)}, h_2, d_{s(h_2)}, \ldots, h_k, d_{s(h_k)}\}
= \{h_1, d_{s(h_1)}\} \sqcup \{h_2, d_{s(h_2)}\} \sqcup \cdots \sqcup \{h_k, d_{s(h_k)}\}
\]
Then by Remark 3.4(1), Claim 5, \( J'' + J' \in J_{0,\ldots,0} \). □

Next, we have;

**Claim 10.** There is \( J'' \in 2^{\{d_1, \ldots, d_{n-1}\}} \) such that \( J'' + J' \in J_{0,\ldots,0} \).

*Proof.* Let \( d_i \in J'' \) with \( n-1 < i \leq N \). Since \( T_G \) is a spanning tree of \( G \), there is a unique cycle in \( T_G \cup \{e_i\} \) (recall that \( e_i \) is the edge corresponding to \( d_i \)). Let \( K_i (\subset \{d_1, \ldots, d_N\}) \) be the \( v \)-cycle corresponding to this cycle. Then since \( K_i \in J_{0,\ldots,0} \) (by Remark 3.4(1), Claim 2), we see that \( \{d_i\} + K_i \subset \{d_1, \ldots, d_{n-1}\} \). Let \( K = \bigcup K_i \). Then we have \( K + J'' \subset \{d_1, \ldots, d_{n-1}\} \). Then \( J'' := K + J'' \) satisfies the condition of Claim 10. □

Let \( \alpha = J + J'' \). Since \( J + J'' = (J + J') + (J' + J'') + (J'' + J'') \), by Claims 8, 9, 10, we see that \( \alpha \in J_{0,\ldots,0} \). Since \( J = J'' + \alpha \) and \( J'' \in 2^{\{d_1, \ldots, d_{n-1}\}} \), we see that \( J \in Y + J_{0,\ldots,0} \), where \( Y = J'' \). This completes the proof of Lemma 3.3. □

As an immediate consequence of the lemma, we have;

\[
2^{V(G)} = \bigcup_{Y \in 2^{\{d_1, \ldots, d_{n-1}\}}} (Y + J_{0,\ldots,0}).
\]

Then we show;

**Proposition 3.6.**

\[
2^{V(G)} = \prod_{Y \in 2^{\{d_1, \ldots, d_{n-1}\}}} (Y + J_{0,\ldots,0}).
\]

*Proof.* We note that for any \( Y \in 2^{\{d_1, \ldots, d_{n-1}\}} \), there exists \( (\epsilon_1, \ldots, \epsilon_{n-1}) \) such that \( Y + J_{0,\ldots,0} \subset J_{\epsilon_1,\ldots,\epsilon_{n-1}} \), since for different elements \( J_1, J_2 \in J_{0,\ldots,0} \), \( \omega(Y + J_1) = \omega(Y + J_2) = \)
$\omega(Y) = (\epsilon_1, \ldots, \epsilon_{n-1})$. Further, we note that $|2^{(d_1, \ldots, d_{n-1})}| = |\{J_{\epsilon_1, \ldots, \epsilon_{n-1}} \mid \epsilon_1, \ldots, \epsilon_{n-1} \in \{0, 1\}\}| = 2^{n-1}$. These together with the above assertions, 

$$2^V(G) = \coprod_{(\epsilon_1, \ldots, \epsilon_{n-1}) \in \mathbb{Z}_2^{n-1}} J_{\epsilon_1, \ldots, \epsilon_{n-1}}$$
and

$$2^V(G) = \bigcup_{Y \in 2^{(d_1, \ldots, d_{n-1})}} (Y + J_{0,0,\ldots,0})$$

show that for each $Y \in 2^{(d_1, \ldots, d_{n-1})}$, there exists unique $(\epsilon_1, \ldots, \epsilon_{n-1})$ with $Y + J_{0,0,\ldots,0} = J_{\epsilon_1, \ldots, \epsilon_{n-1}}$ and that

$$2^V(G) = \prod_{Y \in 2^{(d_1, \ldots, d_{n-1})}} (Y + J_{0,0,\ldots,0}).$$

By Lemma 3.2 and Proposition 3.6,

$$|\mathcal{H}| = |J_{0,0,\ldots,0}| = \frac{|2^V(G)|}{|2^{(d_1, \ldots, d_{n-1})}|} = \frac{2^q}{2^{n-1}} = 2^{q-n+1},$$

where $q$ is the number of the vertices of $G$. On the other hand, by Theorem 2.1, we have;

$$\dim(\text{Im} \varphi) = q - n + 1,$$

hence $|\text{Im} \varphi| = 2^{q-n+1}$. Hence by using Proposition 3.5, we obtain:

$$\mathcal{H} = \text{Im} \varphi.$$

This completes the proof of Theorem 1.3’. Moreover by Lemma 3.2 and Proposition 3.6 again, we obtain the following presentation of the coset decomposition:

$$2^V(G) = \prod_{Y \in 2^{(d_1, \ldots, d_{n-1})}} (Y + \text{Im} \varphi).$$

This completes the proof of Theorem 1.4’.
4. Structure of the Homomorphism $\psi$

In this section, we give proofs of Theorems 1.6 and 1.7. Let $\tilde{G}$, $F(\tilde{G})$, $V(\tilde{G})$, $\varphi$, $\tilde{G}_i$ $(1 \leq i \leq n)$ be as in Section 3. Let $\psi$ be as in Subsection 2.5.

Recall that the definition of the subset $\mathcal{M}$, $\mathcal{H}_2$, $\mathcal{M}_1 \subset 2^{\mathcal{R}(D)}$ in Section 1. We abuse notations to use the same symbols to represent the corresponding subsets of $2^{F(\tilde{G})}$. Namely $\mathcal{M} = \{ M \in 2^{F(\tilde{G})} | \varphi(M) = V(\tilde{G}) \}$, $\mathcal{H}_2 = \{ H \in \ker \varphi | |H| \equiv 0 \bmod 2 \}$ and $\mathcal{M}_1 = \{ M \in \mathcal{M} | |M| \equiv 1 \bmod 2 \}$.

**Proposition 4.1.** Let $\tilde{G}, \varphi, \mathcal{M}$ be as above. Then, there exists $M \in 2^{F(\tilde{G})}$ such that $\varphi(M) = V(\tilde{G})$, i.e., $\mathcal{M} \neq \emptyset$.

**Proof.** Let $V_{ij}$ be as in Subsection 3.2. It is clear that the set of crossings of $D$ admits a representation $V(\tilde{G}_1) \amalg \cdots \amalg V(\tilde{G}_n) \prod_{1 \leq i < j \leq n} V_{ij}$. By Proposition 3.2, there exists a set of regions $H_1$ such that the region crossing change at $H_1$ changes exactly $V(\tilde{G}_1) \amalg \cdots \amalg V(\tilde{G}_n)$. Since $\tilde{G}$ is a projection of a knot diagram on a 2-sphere, it is elementary to show that $|V_{ij}| \equiv 0 \bmod 2$ for each $i, j$ $(1 \leq i < j \leq n)$. Hence by Proposition 3.3, there exists a set of regions $H_{i,j}$ such that the region crossing change at $H_{i,j}$ changes exactly $V_{ij}$ for each $i, j$ $(1 \leq i < j \leq n)$. By taking the sum of the above set of regions, we see that there exists a set of regions $G$ such that the region crossing change at $G$ changes the crossings of $D$ $(= V(\tilde{G}_1) \amalg \cdots \amalg V(\tilde{G}_n) \prod_{1 \leq i < j \leq n} V_{ij})$. $\square$

**Remark 4.1.** Proposition 4.1 together with Propositions 2.1 and 2.2, and Claim 2 shows that $\text{Im} \Psi \subset \text{Im} \Phi$.

Then, we have:

**Lemma 4.1.** $\ker \psi = \mathcal{H}_2 \amalg \mathcal{M}_1$.

**Proof.** Let $J \in \ker \psi$. If $|J| \equiv 0 \bmod 2$, then by Proposition 2.1, $\varphi(J) = \psi(J) = \emptyset$. Hence $J \in \ker \varphi$, and we have $J \in \mathcal{H}_2 \subset \mathcal{H}_2 \amalg \mathcal{M}_1$. If $|J| \equiv 1 \bmod 2$, then by Proposition 2.1, $\varphi(J) = \psi(J) + V(\tilde{G}) = V(\tilde{G})$. Then $J \in \mathcal{M}_1 \subset \mathcal{H}_2 \amalg \mathcal{M}_1$. Hence $\ker \psi \subset \mathcal{H}_2 \amalg \mathcal{M}_1$.

Let $K \in \mathcal{H}_2 \amalg \mathcal{M}_1$. If $|K| \equiv 0 \bmod 2$, then $K \in \mathcal{H}_2$. Then by Proposition 2.1, $\varphi(K) = \psi(K) = \emptyset$. Hence $K \in \ker \psi$. If $|K| \equiv 1 \bmod 2$, then $K \in \mathcal{M}_1$. Then by Proposition 2.1, $\psi(K) = \varphi(K) + V(\tilde{G}) = \emptyset$. Hence $K \in \ker \psi$. Hence $\ker \psi \supset \mathcal{H}_2 \amalg \mathcal{M}_1$. These facts show $\ker \psi = \mathcal{H}_2 \amalg \mathcal{M}_1$. $\square$

We say that $\varphi$ is even type if $\ker \varphi = \mathcal{H}_2$, i.e., each element of $\ker \varphi$ consists of even number of faces.

**Lemma 4.2.** If $\mathcal{M}_1 = \emptyset$, then $\ker \psi = \ker \varphi = \mathcal{H}_2$, particularly $\varphi$ is even type
We first claim that $H \in \ker \varphi$ such that $|H| \equiv 1 \mod 2$. Take and fix an element $M$ of $\mathcal{M}$. Since $\mathcal{M}_1 = \emptyset$, $|M| \equiv 0 \mod 2$. Hence by Remark 2.1, $|M + H| \equiv 1 \mod 2$. However since $H \in \ker \varphi$, $M + H$ is an element of $\mathcal{M}_1$, contradicting the assumption of the lemma. Hence $\ker \varphi = \mathcal{H}_2$. On the other hand, by the assumption $\mathcal{M}_1 = \emptyset$, and Lemma 4.1, we see that $\ker \psi = \mathcal{H}_2$. This completes the proof.

Lemma 4.3. Suppose $\mathcal{M}_1 \neq \emptyset$. Then for each $M \in \mathcal{M}_1$, we have $\mathcal{M}_1 = M + \mathcal{H}_2$. In particular, $|\mathcal{M}_1| = |\mathcal{H}_2|$.

Proof. Let $M' \in \mathcal{M}_1$. Since $\varphi(M' + M) = \varphi(M') + \varphi(M) = V(\tilde{G}) + V(\tilde{G}) = \emptyset$, $M' + M \in \ker \varphi$. On the other hand, since $|M'| \equiv 1 \mod 2$ and $|M| \equiv 1 \mod 2$, $|M' + M| \equiv 0 \mod 2$ by Remark 2.1. Hence $M' + M \in \mathcal{H}_2$. This shows that $M' + M \in \mathcal{H}_2$.

Next let $J \in M + \mathcal{H}_2$. By Remark 2.1, we see that $|J| \equiv 1 \mod 2$. Since $J \in M + \mathcal{H}_2$, $J + M \in \mathcal{H}_2$, hence $\varphi(J + M) = \emptyset$. Since $\varphi(M) = V(\tilde{G})$ and $\varphi(J + M) = \varphi(J) + \varphi(M)$, the fact $\varphi(J + M) = \emptyset$ implies $\varphi(J) = V(\tilde{G})$. Hence $J \in \mathcal{M}$. This fact together with $|J| \equiv 1 \mod 2$ shows $J \in \mathcal{M}_1$.

These show that $\mathcal{M}_1 = M + \mathcal{H}_2$. 

Recall that $\tilde{G}$ can be regarded as the projection of a link diagram. This fact implies that it is enough to show the following theorems for proving Theorems 1.6 and 1.7.

Theorem 1.6' Let $\tilde{G}$, $\varphi$, $\psi$, $\mathcal{H}_2$, $\mathcal{M}$, $\mathcal{M}_1$ be as above. Then we have the following.

(i) If $\varphi$ is even type (i.e., $\ker \varphi = \mathcal{H}_2$), then we have the following.

(a) If $\mathcal{M}_1 = \emptyset$, then $\ker \psi = \ker \varphi (\mathcal{H}_2)$. In particular, $\dim(\ker \psi) = n + 1$.

(b) If $\mathcal{M}_1 \neq \emptyset$, then for any $M \in \mathcal{M}_1$, $\ker \psi = (\ker \varphi) \cap (M + \ker \varphi) = \mathcal{H}_2 \cap (M + \mathcal{H}_2)$. In particular, $\dim(\ker \psi) = n + 2$.

(ii) If $\varphi$ is not even type, then (1) $|\mathcal{H}_2| = \frac{1}{2} |\ker \varphi|$ (hence, $\dim(\mathcal{H}_2) = n$), and (2) $\mathcal{M}_1 \neq \emptyset$ and for any $M \in \mathcal{M}_1$, $\ker \psi = \mathcal{H}_2 \cap (M + \mathcal{H}_2)$. In particular, $\dim(\ker \psi) = n + 1$.

Theorem 1.7' Let $\tilde{G}$, $F(\tilde{G})$, $\varphi$, $\psi$ be as above. Suppose that $\text{Im} \psi \subseteq \text{Im} \varphi$. Let $J \in \text{Im} \varphi$. Then $J \not\in \text{Im} \psi$ if and only if $\exists H \in 2^{F(\tilde{G})}$ s.t. $\varphi(H) = J$ and $|H| \equiv 1 \mod 2$.

In the remainder of this section, we prove Theorems 1.6' and 1.7'.

Lemma 4.4. If $\varphi$ is not even type, then $\mathcal{M}_1 \neq \emptyset$.

Proof. Let $H \in \mathcal{M}$. If $|H| \equiv 1 \mod 2$, then $H \in \mathcal{M}_1$. Then $\mathcal{M}_1 \neq \emptyset$. Suppose that $|H| \equiv 0 \mod 2$. Since $\varphi$ is not even type, there exists an element $B \in \ker \varphi$ such that
|B| \equiv 1 \mod 2. Then H + B \in M. Further |H + B| \equiv 1 \mod 2, by Remark 2.1, hence H + B \in M_1. These show that M_1 \neq \emptyset.

\square

Lemma 4.5.

\[ \dim H_2 = \begin{cases} 
  n + 1 \text{ (if } \varphi \text{ is even type)} \\
  n \text{ (if } \varphi \text{ is not even type)} 
\end{cases} \]

Proof. Suppose that \( \varphi \) is even type, i.e., ker \( \varphi = H_2 \). By (3) in Subsection 2.4, we have \( \dim(H_2) = \dim(\ker \varphi) = n + 1 \).

Suppose that \( \varphi \) is not even type, i.e., ker \( \varphi \neq H_2 \), hence ker \( \varphi \setminus H_2 \neq \emptyset \). Let \( H_1 \in \ker \varphi \setminus H_2 \). Then it is easy to see that ker \( \varphi = H_2 \amalg (H_1 + H_2) \). Hence \( |H_2| = \frac{1}{2}|\ker \varphi| \). This together with (3) implies that \( \dim(H_2) = n \).

\square

Lemma 4.6. If \( \varphi \) is even type and \( M_1 \neq \emptyset \), then \( M = M_1 \).

Proof. Suppose that there exists \( M \in M \) such that \( |M| \equiv 0 \mod 2 \). Let \( M' \in M_1 \). Then since \( \varphi(M' + M) = V(\tilde{G}) + V(\tilde{G}) = \emptyset \), \( M' + M \in \ker \varphi \). On the other hand, by Remark 2.1, we see that \( |M' + M| \equiv 1 \mod 2 \), contradicting the assumption that \( \varphi \) is even type. Hence \( M = M_1 \).

Now we give a proof of Theorem 1.6'. The proof is divided into the following cases.

**Case i:** \( \varphi \) is even type.

This case is divided into the following cases.

**Case i-a:** \( M_1 = \emptyset \).

Since \( M_1 = \emptyset \), by Lemma 4.2, we see that \( \ker \psi = H_2 \). This gives the conclusion (i)-(a) of Theorem 1.6'.

**Case i-b:** \( M_1 \neq \emptyset \).

Take and fix an element \( M \) of \( M_1 \). By Lemmas 4.1 and 4.3, \( \ker \psi = H_2 \amalg M_1 = H_2 \amalg (M + H_2) \). Since \( \dim(H_2) = n + 1 \) by Lemma 4.5, this shows that \( \dim(\ker \psi) = n + 2 \). This gives the conclusion (i)-(b) of Theorem 1.6'.

**Case ii:** \( \varphi \) is not even type.

By the condition of Case ii and Lemma 4.4, \( M_1 \neq \emptyset \). Then take and fix an element \( M \in M_1 \). By Lemma 4.1, we see that \( \ker \psi = H_2 \amalg M_1 \). Since \( M_1 = M + H_2 \) by Lemma 4.3, \( \ker \psi = H_2 \amalg (M + H_2) \). Hence \( \dim(\ker \psi) = \dim(H_2) + 1 \). Here by Lemma 4.5, \( \dim(H_2) = n \). Therefore we see that \( \dim(\ker \psi) = n + 1 \). This gives the conclusion (ii) of Theorem 1.6'.

These complete the proof of Theorem 1.6'.
Next we give a proof of Theorem 1.7. By Corollary 1.2, the assumption of Theorem 1.7 is rephrased as follows: “φ is even type and ∃M ∈ M s.t. |M| ≡ 1 mod 2.”

**Proof of “only if part of Theorem 1.7”**. Suppose that J ∈ Imφ \ Imψ. Since J ∈ Imφ, ∃H s.t. φ(H) = J. If |H| ≡ 0 mod 2, then by Proposition 2.1 ψ(H) = φ(H) = J, contradicting J ∈ Imφ \ Imψ. Hence |H| ≡ 1 mod 2.

**Proof of “if part of Theorem 1.7”**. Suppose that ∃H ∈ 2F(Ḡ) s.t. φ(H) = J and |H| ≡ 1 mod 2. Suppose for a contradiction that ∃H′ ∈ 2F(Ḡ) s.t. ψ(H′) = J. Here we claim that |H′| ≡ 1 mod 2. In fact, if |H′| ≡ 0 mod 2, then by Proposition 2.1 ψ(H′) = φ(H′) = J. Hence φ(H + H′) = J + J = ∅, i.e., H + H′ ∈ ker φ. On the other hand, by Remark 2.1, we see that |H + H′| ≡ 1 mod 2. This contradicts the condition that φ is even type and this shows that |H′| ≡ 1 mod 2. Since |H′| ≡ 1 mod 2, by Proposition 2.1 φ(H′) = ψ(H′) + V(Ḡ) = J + V(Ḡ). Then we have φ(H + H′) = φ(H) + φ(H′) = V(Ḡ), hence H + H′ ∈ M. By Remark 2.1, we see that |H + H′| ≡ 0 mod 2. Hence H + H′ ∈ M \ M1. On the other hand, since Imψ ⊇ Imφ, φ is even type and M = M1 by Corollary 1.2, a contradiction. This completes the proof.

**A basis of ker ψ for irreducible ḡ**

Let ḡ, φ, ψ as above. Let D be a link diagram such that |D| = ḡ, and c a crossing of D. We note that φ is even type if and only if there exists a basis of ker φ such that each element of the basis consists of even number of faces. Hence, according to Theorem 1.2', we have the next assertion.

**Assertion 4** Suppose that ḡ is irreducible. Then, φ is even type if and only if each |Bi| (0 ≤ i ≤ n) is an even number.

Suppose that the conclusion (i)-(a) of Theorem 1.6' holds. Then we have ker ψ = ker φ. Hence, by Theorem 1.2', we have:

**Assertion 5** Suppose that ḡ is irreducible. If the conclusion (i)-(a) of Theorem 1.6' holds, then {B0, B1, ..., Bn} is a basis of ker ψ.

Then we show the next assertion.

**Assertion 6** Suppose that ḡ is irreducible. If the conclusion (i)-(b) of Theorem 1.6' holds, then for any M ∈ M1, {B0, B1, ..., Bn, M} is a basis of ker ψ.

**Proof**: By Theorem 1.6', {B0, B1, ..., Bn} is a basis of ker φ. Since φ is even type, ker φ = H2. Take and fix an element M of M1. By Proposition 4.1 and 4.3, ker ψ = H2II(M + H2). This fact shows that {B0, B1, ..., Bn, M} generates ker ψ.
Since \( \dim(\ker \psi) = n + 2 \) (Lemma 4.5) and \( \{|B_0, B_1, \ldots, B_n, M|\} = n + 2 \), we see that \( \{B_0, B_1, \ldots, B_n, M\} \) is a basis of \( \ker \psi \).

\[\square\]

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REFERENCES