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Jump measure densities corresponding to Brownian motion on an annulus

TAKEMURA Tomoko*

1 Introduction

We consider jump measure densities for Dirichlet forms of non-local type corresponding to skew product diffusion process of a one dimensional diffusion process on \(\mathbb{R}\) and the spherical Brownian motion on \(S^{d-1}\). We showed a limit theorem for Dirichlet forms of local type to that of non-local type, in view of semi groups for time changes of these skew product in [7]. The Dirichlet forms corresponding to the limit processes are obtained in [8]. The Dirichlet form corresponding to the limit process has a diffusion part, a jump part, and a killing part. We are interested characteristics of a one dimensional diffusion process and the spherical Brownian motion appeared in the jump part of the Dirichlet form. In this paper we discuss a jump rate corresponding to time changed skew product diffusion process of an extended Bessel process and the spherical Brownian motion. We focus on 2 dimensional case so that the corresponding skew product diffusion processes is recurrent. We can find the effects of recurrent property to jump rates. We clarify jump measure densities corresponding to Brownian motion on annulus.

2 Preliminaries

Let \(s^R\) be a continuous strictly increasing function on an open interval \(I = (l_1, l_2)\), and \(m^R\) be a right continuous nondecreasing function on \(I\), where \(0 \leq l_1 < l_2 \leq \infty\). We denote by \(R = [R(t), P^R_t]\) a one dimensional diffusion process on \(I\) with scale function \(s^R\), speed measure \(m^R\) and no killing measure. We denote by \(ds^R\) and \(dm^R\) the measures induced by \(s^R\) and \(m^R\), respectively. We assume that \(\text{supp}[m^R]\), the support of \(dm^R\), coincides with \(I\). We consider the following symmetric bilinear form \((\mathcal{E}^R, \mathcal{F}^R)\).

\[
\mathcal{E}^R(u, v) = \int_I \frac{du}{ds^R} \frac{dv}{ds^R} ds^R,
\]

(2.1)

\[
\mathcal{F}^R = \{ u \in L^2(I, m^R) : u \text{ is absolutely continuous on } I \text{ with respect to } ds^R \text{ and } \mathcal{E}^R(u, u) < \infty, u(l_i) = 0 \text{ if } |s^R(l_i)| < \infty \}.
\]

(2.2)

We set \(\mathcal{C}^R = \{ u \circ s^R : u \in C^1_0(J) \}\), where \(J = s^R(I)\) and \(C^1_0(J)\) is the set of all continuously differentiable functions on \(J\) with compact support. Then \((\mathcal{E}^R, \mathcal{F}^R)\) is a regular, strongly local, irreducible Dirichlet form on \(L^2(I, m^R)\) possessing \(\mathcal{C}^R\) as its core and corresponding to the one dimensional diffusion \(R = [R_t, P^R_t]\) (see [1], [4]). We note that the one dimensional diffusion \(R\) has the local time \(l^R(t, r)\) which is continuous with respect to \((t, r) \in [0, \infty) \times I\) and satisfies \(\int_0^t 1_A(R(u)) \, du = \int_A l^R(t, r) \, dm^R(r)\), \(t > 0\), for every measurable set \(A \subset I\) (see [5]), where \(1_A\) is the indicator for a set \(A\). We set

\[
J_{\mu, \nu}(l_i) = \int_{(c, l_i)} d\mu(x) \int_{(c, x]} d\nu(y), \text{ an arbitrary point } c \in I,
\]

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for Borel measures $\mu$ and $\nu$ on $I$. Following [2], we call the end point $l_i$ to be

- $(s,m)$-regular if $J_{s,m}(l_i) < \infty$ and $J_{m,s}(l_i) < \infty$,
- $(s,m)$-exit if $J_{s,m}(l_i) < \infty$ and $J_{m,s}(l_i) = \infty$,
- $(s,m)$-entrance if $J_{s,m}(l_i) = \infty$ and $J_{m,s}(l_i) < \infty$,
- $(s,m)$-natural if $J_{s,m}(l_i) = \infty$ and $J_{m,s}(l_i) = \infty$,

where $s = s^R$ and $m = m^R$. Following [4], we call $\mathcal{E}^R$ to be conservative if $p_t^R1 = 1$, $t > 0$. Since $p_t^R1(r) = P^R_t(t < \sigma^R_1 \wedge \sigma^R_2)$, we see that $p_t^R1 = 1$ if and only if

\begin{equation}
\text{(2.3)} \quad \text{both of } l_i, i = 1, 2, \text{ are } (s^R, m^R)\text{-entrance or natural,}
\end{equation}

where $\sigma^X_t$ is a first hitting time to $\xi$ of a process $X = [X_t, P^X_t]$, that is, $\sigma^X_t = \inf\{t > 0; X_t = \xi\}$, and $a \wedge b$ is the smaller of $a$ and $b$.

Next we consider the spherical Brownian motion $BM(S^d)$ on $S^d \subset \mathbb{R}^{d+1}$ with generator $\frac{1}{2} \Delta$, where $\Delta$ is the spherical Laplacian on $S^d$. We denote by $p_t^\Theta$ the semigroup of the spherical Brownian motion $\Theta = [\Theta(t), P^\Theta_t]$, that is,

\[ p_t^\Theta f(\theta) = E^{P^\Theta_t}[f(\Theta(t))] = \int_{S^{d-1}} p_t^\Theta(t, \theta, \varphi)f(\varphi)dm^\Theta(\varphi), \quad t > 0, \theta \in S^{d-1}, \]

for $f \in C_b(S^{d-1})$, where $p_t^\Theta(t, \theta, \varphi)$ stands for the transition probability density of $\Theta$. It is known that $p_t^\Theta(t, \theta, \varphi)$ is represented by spherical harmonics $S_n^l$, that is,

\begin{equation}
\text{(2.4)} \quad p_t^\Theta(t, \theta, \varphi) = \sum_{n=0}^{\infty} e^{-\gamma_n t} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi),
\end{equation}

where $\gamma_n = \frac{1}{2} n(n + d - 2)$, $\kappa(n) = (2n + d - 2) \cdot (n + d - 3)! / n!(d - 2)!$ which is the number of spherical harmonics of weight $n$, $\frac{1}{2} \Delta S_n^l = -\gamma_n^l S_n^l$, and

\[ \int_{S^{d-1}} S_n^l S_m^k dm^\Theta = \begin{cases} l, & l = k; \\ 0, & l \neq k. \end{cases} \]

Since $\int_{S^{d-1}} dm^\Theta = 2\pi^{\frac{d}{2}} / \Gamma(\frac{d}{2})$, which is the total area of the spherical surface $S^{d-1}$, we see $S_0^d = \{2\pi^{\frac{d}{2}} / \Gamma(\frac{d}{2})\}^{-1/2}$. Note that $\kappa(0) = 1$. When $d = 2$, (2.4) is reduced to

\[ p_t^\Theta(t, \theta, \varphi) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{-\frac{\gamma_n}{2} t} \{\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi\} = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} e^{-\frac{\gamma_n}{2} t} \cos n(\theta - \varphi). \]

Let $\nu$ be a Radon measure on $I$ and assume that $\text{supp}[\nu]$, the support of $\nu$, coincides with $I$. We set

\[ f(t) = \int_I I^R(t, r) d\nu(r), \quad t \geq 0. \]

Since $\text{supp}[\nu] = I$, we see that

\[ P_t^R(f(t) > 0, t > 0) = 1, \quad r \in I. \]
We assume (2.3). Let $Y = [Y(t) = (R(t), \Theta(f(t)))], P^Y_{(r, \theta)} = P^R \otimes P^\Theta_{(r, \theta)}, (r, \theta) \in I \times S^{d-1}$ be the skew product of the one dimensional diffusion process $R$ and the spherical Brownian motion $\Theta$ with respect to the positive continuous additive functional $f(t)$, and set

$$
\mathcal{E}^Y(f, g) = \int_{S^{d-1}} \mathcal{E}^R(f(\cdot, \theta), g(\cdot, \theta)) dm^\Theta(\theta) + \int_I \mathcal{E}^\Theta(f(\cdot, \cdot), g(\cdot, \cdot)) d\nu(r),
$$

for $f, g \in \mathcal{C}^Y$, where

$$
\mathcal{C}^Y = \{f(s^R(r), \theta) : f \in C^\infty(J \times S^{d-1})\},
$$

and $J = s^R(I)$. The form $(\mathcal{E}^Y, \mathcal{C}^Y)$ is closable on $L^2(I \times S^{d-1}, m^R \otimes m^\Theta)$. The closure $(\mathcal{E}^Y, \mathcal{F}^Y)$ is a regular Dirichlet form and it is corresponding to the skew product $Y$.

Let $\mu$ be a non-trivial Radon measure on $I$ and set

$$
g(t) = \int_I \tau^R(t, r) d\mu(r), \quad t > 0.
$$

We denote by $\tau(t)$ the right continuous inverse of $g(t)$. We consider the time changed process $\Xi = [\Xi_t = (R_\tau(t), \Theta(\tau(t))), P^\Xi_{(r, \theta)} = P^R \otimes P^\Theta_{(r, \theta)}, (r, \theta) \in I \times S^{d-1}]$. Note that the time changed process $U = [R_\tau(t), P^R]$ is a one dimensional diffusion process with the scale function $s^R$ and the speed measure $\mu$. We set $A = \text{supp}[\mu]$ and $\Gamma = \Lambda \times S^{d-1}$. Also note that the time changed process $\Xi$ is essentially defined on $\Gamma$. In the following, besides (2.3) we assume that

for any compact set $B \subset I$, there exists a positive

$$
\text{(2.5)} \quad \text{constant } M_B \text{ satisfying } 1_B(r) d\sigma^R(r) \leq M_B 1_B(r) d\sigma^R(r).
$$

We note that the measure $\mu \otimes m^\Theta$ charges no set of zero $\mathcal{E}^Y$-capacity.

We note that $I \setminus A = \bigcup_{k \in K} I_k$, a finite or a countable disjoint union of open intervals $I_k = (a_k, b_k)$ with the end points belonging to $\Lambda \cup \{l_1, l_2\}$. Since $\mathcal{C}^Y|_\Gamma$ is a core of $(\mathcal{E}^\Xi, \mathcal{F}^\Xi)$, we fix a $u \in \mathcal{C}^Y$ and set $f = u|_\Gamma$. Then $f \in \mathcal{F}^\Xi$ and

$$
\mathcal{E}^\Xi(f, f) = \mathcal{E}^Y(H_f u, H_f u),
$$

where $H_f u(r, \theta) = E_F^{R(\cdot, \theta)} \left[u \left(Y_{\sigma^\Xi_f}; \sigma^\Xi_f < \infty\right)\right]$, and $\sigma^\Xi_f = \inf\{t > 0 : Y_t \in \Gamma\}$. We are going to derive an explicit form of $\mathcal{E}^\Xi_{|_H}(H_f u, H_f u)$. For $r \in I_k = (a_k, b_k)$ and $\theta, \varphi \in S^{d-1}$, we set

$$
\text{(2.6)} \quad G_{k,1}(r; \theta, \varphi) = E^{P^R}_{\sigma^R_{a_k}} \left[p^\Theta(\sigma^R_{b_k}, \theta, \varphi); \sigma^R_{a_k} < \sigma^R_{b_k}\right],
$$

$$
\text{(2.7)} \quad G_{k,2}(r; \theta, \varphi) = E^{P^R}_{\sigma^R_{a_k}} \left[p^\Theta(\sigma^R_{a_k}, \theta, \varphi); \sigma^R_{a_k} < \sigma^R_{b_k}\right].
$$

By means of (2.4) we see that

$$
G_{k,1}(r; \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S^l_n(\theta) S^l_n(\varphi) E^{P^R}_{\sigma^R_{a_k}} \left[e^{-\gamma_n f(\sigma^R_{a_k})}; \sigma^R_{a_k} < \sigma^R_{b_k}\right],
$$

$$
G_{k,2}(r; \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S^l_n(\theta) S^l_n(\varphi) E^{P^R}_{\sigma^R_{a_k}} \left[e^{-\gamma_n f(\sigma^R_{a_k})}; \sigma^R_{a_k} < \sigma^R_{b_k}\right],
$$

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for \( r \in I_k = (a_k, b_k) \) and \( \theta, \varphi \in S^{d-1} \). By virtue of a general theory of one dimensional diffusion’s, there exist the following limits (see [6]).

\[
\begin{align*}
J_k^{1,2}(\theta, \varphi) &= \lim_{r \to 0} D_{s_n(r)} G_{k,2}(r; \theta, \varphi), \\
J_k^{2,2}(\theta, \varphi) &= \lim_{r \to 0} D_{s_n(r)} G_{k,1}(r; \theta, \varphi), \\
J_k^{1,1}(\theta, \varphi) &= -\lim_{r \to 0} D_{s_n(r)} G_{k,2}(r; \theta, \varphi), \\
J_k^{2,1}(\theta, \varphi) &= -\lim_{r \to 0} D_{s_n(r)} G_{k,1}(r; \theta, \varphi).
\end{align*}
\]

We denote by \( \mathcal{M} \) the product measure \( m^\Theta \otimes m^\Theta \). Assume that \( \int_\Lambda ds^R > 0 \). Let \( u \in \mathcal{C}^1 \Gamma \) and put \( f = u|_\Gamma \). Then there exists the limit

\[
\frac{\partial^*_R f(r, \theta)}{r} = \lim_{r'(\in \Gamma) \to r} \frac{f(r', \theta) - f(r, \theta)}{s^R(r') - s^R(r)} = \lim_{r' \to r} \frac{u(r', \theta) - u(r, \theta)}{s^R(r') - s^R(r)},
\]

for \( ds^R \)-a.e. \( r \in \Lambda \) and every \( \theta \in S^{d-1} \).

**Theorem 2.1** ([7]) Assume \( \Lambda \neq I \), (2.3) and (2.5). Then the Dirichlet form \( (\mathcal{E}_\Xi, \mathcal{F}_\Xi) \) of \( \Xi \) is regular on \( L^2(\Gamma, \mu \otimes m^\Theta) \) and has \( \mathcal{C}^1 |_\Gamma \) as a core. For \( f \in \mathcal{C}^1 |_\Gamma \), the Dirichlet form \( (\mathcal{E}_\Xi, \mathcal{F}_\Xi) \) is given by the following.

\[
\begin{align*}
\mathcal{E}_\Xi(f, f) &= \int_{\Gamma} \partial^*_R f(r, \theta)^2 ds^R(r) dm^\Theta(\theta) + \int_{\Lambda} \mathcal{E}^\Theta(f(r, \cdot), f(r, \cdot)) d\nu(r) \\
&+ \frac{1}{2} \sum_{k \geq K: \Sigma \leq k \leq \underline{K}} \int_{S^{d-1} \times S^{d-1}} \left( f(a_k, \theta) - f(a_k, \varphi) \right)^2 J_k^{1,1}(\theta, \varphi) dM(\theta, \varphi) \\
&+ \frac{1}{2} \sum_{k \geq K: \Sigma \leq k \leq \underline{K}} \int_{S^{d-1} \times S^{d-1}} \left( f(b_k, \theta) - f(b_k, \varphi) \right)^2 J_k^{2,2}(\theta, \varphi) dM(\theta, \varphi) \\
&+ \frac{1}{2} \sum_{k \geq K: \Sigma \leq k \leq \underline{K}} \int_{S^{d-1} \times S^{d-1}} \left( f(a_k, \theta) - f(b_k, \varphi) \right)^2 J_k^{1,2}(\theta, \varphi) dM(\theta, \varphi) \\
&+ \frac{1}{2} \sum_{k \geq K: \Sigma \leq k \leq \underline{K}} \int_{S^{d-1} \times S^{d-1}} \left( f(b_k, \theta) - f(a_k, \varphi) \right)^2 J_k^{2,1}(\theta, \varphi) dM(\theta, \varphi).
\end{align*}
\]

Here the first term of the right hand side vanishes in case that \( \int_\Lambda ds^R(r) = 0 \).

## 3 Exact representation of Jump measure densities

In this section we consider the jump measure densities associated with Brownian motion on annulus. Let \( Y \) be a skew product of an extended Bessel process and the spherical Brownian motion, and \( X \) be the time changed process of \( Y \). Here an extended Bessel process means that the diffusion process on \( I \) coincides with a Bessel process on subinterval. We give an exact representation of jump measure densities on annulus for \( X \). First we summarize some properties of a Bessel process. Let \( d = 2 \) and \( R \) be the Bessel process on \( I = (0, \infty) \) with the generator \( G^R = \frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \). We may set \( ds^R(r) = 2r^{-1}dr \) and \( dm^R(r) = rdr \). By means of (2.1) and (2.2), the Dirichlet form associated with \( R \) is given by

\[
\mathcal{E}^R(u, v) = \int_0^\infty \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} dr,
\]
\( \mathcal{F}^R = \{ u \in L^2(I, rdr) : u \text{ is absolutely continuous on } (0, \infty) \text{ with respect to } dr \text{ and } \mathcal{E}^R(u, u) < \infty \} \).

We note that \((\mathcal{E}^R, \mathcal{F}^R)\) has \(C^1_0(I)\) as its core, for \(C^1_0(I) = \mathcal{C}^R = \{ u \circ s^R, u \in C^1_0(J) \}\) with \(J = s^R(I)\). The end point 0 is \((s^R, m^R)\)-entrance and the end point \(\infty\) is \((s^R, m^R)\)-natural.

We set
\[
f(t) = \int_I l^R(t, r) r^{-1} dr = \int_0^t R_u^{-2} du, \quad t > 0.
\]

In the same way as in [5], we obtain the following
\[
\begin{align*}
E^R_{[\gamma_a, \gamma_b]}( \sigma_a^R, \sigma_b^R) & = \frac{(b/a)^n - (r/b)^n}{(b/a)^n - (a/b)^n}, \quad a < r < b. \\
E^R_{[\gamma_a, \gamma_b]}( \sigma_b^R, \sigma_a^R) & = \frac{(a/r)^n - (b/a)^n}{(b/a)^n - (a/b)^n}, \quad a < r < b.
\end{align*}
\]

Here \(\gamma_n = \frac{1}{2} n^2, \quad 0 < a < b < \infty\) and \(n \geq 0\). If \(n = 0\), (3.1) and (3.2) are reduces to (3.3) and (3.4), respectively.

\[
\begin{align*}
P^R_{\gamma_a} (\sigma_a^R, \sigma_b^R) & = \frac{\log b/r}{\log b/a}, \quad a < r < b, \\
P^R_{\gamma_b} (\sigma_a^R, \sigma_b^R) & = \frac{\log r/a}{\log b/a}, \quad a < r < b.
\end{align*}
\]

In the following we discuss the Brownian motion on an annulus \((a, b) \times S^1\).

We denote by \(R^e\) an extended Bessel process on \(I\) and by \(\Theta\) the spherical Brownian motion on \(S^1\). We set
\[
X = \{ X(t) = (R^e_{\tau(t)}, \Theta_{\tau(t)}) \}, \quad P^\infty_{(r, \Theta)} = P^R_{\gamma_a} \otimes P^{\Theta}_{\gamma_b}, \quad (r, \Theta) \in I \times S^{d-1},
\]
where \(\tau(t)\) is the right continuous inverse of \(g(t) = \int_I l^R(t, r) d\mu(r), t > 0\) and \(f(t) = \int_I l^R(t, r) d\nu(r), t > 0\). Let \(d\mu(r) = 1_{(0, a), (b, \infty)}(r) dm^R(r)\) and \(d\nu(r) = 1_{(0, a)}(r) d\omega(r) + 1_{(a, b)}(r) r^{-2} dm^R(r) + 1_{(b, \infty)}(r) d\omega(r), \) where \(0 < a < b < \infty\) and \(\omega\) is a Radon measure on \(I\) such that \(\|\omega\| = I\) and \(\int_{(0, a), (b, \infty)} s^R(r) d\omega(r) = \infty\). This one dimensional diffusion process \(R^e\) on \((0, \infty)\) coincides with the Bessel process \(R\) on subinterval \((a, b)\). By virtue of Theorem 2.1, we get the following. For \(f \in \mathcal{C}^V_{(0, a), (b, \infty) \times S^1} = C^\infty_0(I \times S^1)\),
\[
\mathcal{E}^X(f, f) = \frac{1}{2} \int_{(0, a), (b, \infty) \times S^1} \frac{\partial f}{\partial r}(r, \theta)^2 rdr \, d\theta
\]
\[
+ \frac{1}{2} \int_{(0, a), (b, \infty) \times S^1} \mathcal{E}^\Theta(f(r, \cdot), f(r, \cdot)) \, d\omega(r)
\]
\[
+ \frac{1}{2} \int_{S^1 \times S^1} \{ f(a, \theta) - f(a, \varphi) \}^2 J_{1,1}(\theta, \varphi) \, d\mathcal{M}(\theta, \varphi)
\]
\[
+ \frac{1}{2} \int_{S^1 \times S^1} \{ f(b, \theta) - f(b, \varphi) \}^2 J_{2,2}(\theta, \varphi) \, d\mathcal{M}(\theta, \varphi)
\]
\[
+ \frac{1}{2} \int_{S^1 \times S^1} \{ f(a, \theta) - f(b, \varphi) \}^2 J_{1,2}(\theta, \varphi) \, d\mathcal{M}(\theta, \varphi)
\]
\[
+ \frac{1}{2} \int_{S^1 \times S^1} \{ f(b, \theta) - f(a, \varphi) \}^2 J_{2,1}(\theta, \varphi) \, d\mathcal{M}(\theta, \varphi),
\]
where $G_1$ and $G_2$ are given by (2.6) and (2.7) with $a = a_k$ and $b = b_k$, respectively, and $J^{(k)}$ are given by $J^{(k)}$. We obtain the jump measure densities as follows. By means of an infinite series, we have

$$E^{R^1}[e^{-\gamma \sigma^R}; \sigma^R < \sigma^R] = \frac{(b/r)^n - (r/b)^n}{(b/a)^n - (a/b)^n}$$

$$= \sum_{m=0}^{\infty} \left\{ \left( \frac{a}{r} \right)^n \left( \frac{a}{b} \right)^{2mn} - \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn+n} \right\},$$

$$E^{R^2}[e^{-\gamma \sigma^R}; \sigma^R < \sigma^R] = \frac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^n}$$

$$= \sum_{m=0}^{\infty} \left\{ \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn} - \left( \frac{a}{r} \right)^n \left( \frac{a}{b} \right)^{2mn+n} \right\}.$$

Since $|S_n(\theta)| < 1$ and d’Alembert principle, we obtain

$$G_1(r; \theta, \varphi) = \sum_{n=1}^{\infty} \sum_{l=1}^{k(n)} S_n^l(\theta) S_n^l(\varphi) \sum_{m=0}^{\infty} \left\{ \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn} - \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn+n} \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{l=1}^{k(n)} S_n^l(\theta) S_n^l(\varphi) \frac{\log r/a}{\log b/a} + \sum_{n=1}^{\infty} \sum_{l=1}^{k(n)} S_n^l(\theta) S_n^l(\varphi) \sum_{m=0}^{\infty} \left\{ \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn} - \left( \frac{r}{b} \right)^n \left( \frac{a}{b} \right)^{2mn+n} \right\}$$

$$= \frac{1}{2\pi \log b/a} \left\{ \frac{1}{\pi} - 2 \left( \frac{r}{b} \right)^{2m} \cos(\theta - \varphi) + \left( \frac{r}{b} \right)^{2m} \cos(\theta + \varphi) + \left( \frac{r}{b} \right)^{2m} \cos(\theta - \varphi) + \left( \frac{r}{b} \right)^{2m} \cos(\theta + \varphi) \right\}.$$

Therefore

$$D_{\mu^0}g_1(r; \theta, \varphi) = -\frac{1}{4\pi \log a/b}$$

$$+ \sum_{m=0}^{\infty} \left\{ \frac{1}{2\pi} - 2 \left( \frac{r}{b} \right)^{2m} \cos(\theta - \varphi) - \left( \frac{r}{b} \right)^{2m} \cos(\theta - \varphi) + \left( \frac{r}{b} \right)^{2m} \cos(\theta + \varphi) + \left( \frac{r}{b} \right)^{2m} \cos(\theta + \varphi) \right\}.$$

$$+ \frac{1}{\pi} \left\{ \frac{1}{2\pi} - 2 \left( \frac{r}{b} \right)^{2m+1} \cos(\theta - \varphi) - \left( \frac{r}{b} \right)^{2m+1} \cos(\theta - \varphi) + \left( \frac{r}{b} \right)^{2m+1} \cos(\theta + \varphi) + \left( \frac{r}{b} \right)^{2m+1} \cos(\theta + \varphi) \right\}.$$
We note that

\[ J = \frac{1}{4 \pi} \log a/b - \sum_{m=0}^{\infty} \left[ \frac{1}{2 \pi} \right] \left( \frac{a}{b} \right)^{2m} \cos(\theta - \varphi) + \left( \frac{1}{a} \right)^{2m} \cos(\theta + \varphi) \]

\[ + \frac{1}{\pi} \left\{ \left( \frac{a}{b} \right)^{2m} \cos(\theta - \varphi) - \left( \frac{a}{b} \right)^{4m} \right\}^2 + \frac{1}{2 \pi} \left[ 1 - 2 \left( \frac{a}{b} \right)^{2m+1} \cos(\theta - \varphi) - \left( \frac{a}{b} \right)^{4m+1} \right] \]

Therefore we obtain the jump rates.

\[ J^{1,1}(\theta, \varphi) = \lim_{r \to a} D_{a, \psi(r)} G_{k,2}(r; \theta, \varphi) \]

\[ J^{1,2}(\theta, \varphi) = -\frac{1}{4 \pi} \log a/b + \sum_{m=0}^{\infty} \left[ \frac{1}{2 \pi} \right] \left( \frac{a}{b} \right)^{2m+1} \cos(\theta - \varphi) + \left( \frac{1}{a} \right)^{2m+1} \cos(\theta + \varphi) \]

\[ + \frac{2}{\pi} \left\{ \left( \frac{a}{b} \right)^{2m+1} \cos(\theta - \varphi) - \left( \frac{a}{b} \right)^{4m+2} \right\}^2 \]

\[ J^{2,1}(\theta, \varphi) = J^{1,2}(\theta, \varphi), \]

\[ J^{2,2}(\theta, \varphi) = J^{1,1}(\theta, \varphi). \]

We note that \( J^{1,1}(\theta, \varphi), J^{1,2}(\theta, \varphi), J^{2,1}(\theta, \varphi), \) and \( J^{2,2}(\theta, \varphi) \) means the jump rate to \( a \times S^1 \) from \( a \times S^1 \), to \( b \times S^1 \) from \( a \times S^1 \), to \( a \times S^1 \) from \( b \times S^1 \), and to \( b \times S^1 \) from \( b \times S^1 \), respectively.

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4 Remarks

In this section we give some remarks. First we show that the relation between the jump rate on an annulus and that on a disc.

Lemma 4.1 For $\theta - \varphi \neq 2n\pi$, we have the limit of the jump rates as $a$ goes to 0 and $b$ goes to $\infty$.

\begin{equation}
\lim_{a \to 0} J^{1,1}(\theta, \varphi) = \lim_{b \to \infty} J^{1,1}(\theta, \varphi) = \lim_{a \to 0} J^{2,2}(\theta, \varphi) = \lim_{b \to \infty} J^{2,2}(\theta, \varphi) = \frac{1}{4\pi} \frac{1}{1 - \cos(\theta - \varphi)},
\end{equation}

\begin{equation}
\lim_{a \to 0} J^{1,2}(\theta, \varphi) = \lim_{b \to \infty} J^{1,2}(\theta, \varphi) = \lim_{a \to 0} J^{2,1}(\theta, \varphi) = \lim_{b \to \infty} J^{2,1}(\theta, \varphi) = 0.
\end{equation}

The equations (4.1) equal to the jump rate on disk obtained Example in [7]. Especially we note that \(\lim_{b \to \infty} J^{2,2}(\theta, \varphi)\) means the jump rate at $\infty$. The equations (4.2) suggest the effects of the recurrence property and entrance at the end point 0.

Next we note jump rates in the special case of argument. We note that the following representation.

\[
J^{1,1}(\theta, \varphi) = \frac{1}{4\pi \log a/b} \left[ \frac{1}{2\pi} \frac{1}{2(1 - \cos(\theta - \varphi))} + \frac{1}{4\pi} \right] \\
+ \frac{1}{2\pi} \left\{ \frac{2}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} + \frac{1}{\pi} \left\{ \frac{2}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} \right\} \right\}^2 \\
- \sum_{m=1}^{\infty} \left\{ \frac{1}{2\pi} \frac{2m}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} + \frac{1}{\pi} \left\{ \frac{2m}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} \right\} \right\}
\]

\[
J^{1,2}(\theta, \varphi) = \\
- \frac{1}{4\pi \log a/b} + \frac{1}{\pi} \left\{ \frac{2}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} + \frac{1}{\pi} \left\{ \frac{2}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} \right\} \right\} \right\}^2 \\
+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\pi} \frac{2m+1}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} + \frac{2}{\pi} \left\{ \frac{2m+1}{1 - 2\left(\frac{a}{b}\right) \cos(\theta - \varphi)} \right\} \right\}
\]

Lemma 4.2 We have the following limits as $\theta - \varphi$ goes to $2n\pi$.

\[
\lim_{\theta - \varphi \to 2n\pi} J^{i,j}(\theta, \varphi) \times (1 - \cos(\theta - \varphi)) = \frac{1}{4\pi},
\]

\[
\lim_{\theta - \varphi \to 2n\pi} J^{i,j}(\theta, \varphi) \times (1 - \cos(\theta - \varphi)) = 0.
\]

Finally we discuss the special case except $\theta - \varphi = 2n\pi$. 

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Lemma 4.3 Let \( n \in \mathbb{Z} \) and \( i, j = 1, 2 \) and \( i \neq j \).

(i) In the case of \( \theta - \varphi = \frac{1}{2} \pi + n \pi \), that is, \( \cos(\theta - \varphi) = 0 \).

\[
J_{i,i}(\theta, \varphi) = \frac{1}{4 \pi} \frac{1}{\log a/b} + \sum_{m=0}^{\infty} \frac{1}{\pi} \left[ \frac{\left( \frac{a}{b} \right)^{4m}}{(1 + \left( \frac{a}{b} \right)^{4m})^2} + \frac{\left( \frac{a}{b} \right)^{4m+4}}{(1 + \left( \frac{a}{b} \right)^{4m+4})^2} \right],
\]

\[
J_{i,j}(\theta, \varphi) = -\frac{1}{4 \pi} \frac{1}{\log a/b} - \sum_{m=0}^{\infty} \frac{1}{\pi} \left( \frac{2}{\left( \frac{a}{b} \right)^{4m+2}} \right)^{\frac{1}{2}}.
\]

(ii) In the case of \( \theta - \varphi = (2n + 1) \pi \), that is, \( \cos(\theta - \varphi) = -1 \).

\[
J_{i,i}(\theta, \varphi) = \frac{1}{4 \pi} \frac{1}{\log a/b} + \sum_{m=0}^{\infty} \frac{1}{\pi} \left[ \frac{\left( \frac{a}{b} \right)^{2m}}{(1 + \left( \frac{a}{b} \right)^{2m})^2} + \frac{\left( \frac{a}{b} \right)^{2m+2}}{(1 + \left( \frac{a}{b} \right)^{2m+2})^2} \right],
\]

\[
J_{i,j}(\theta, \varphi) = -\frac{1}{4 \pi} \frac{1}{\log a/b} - \sum_{m=0}^{\infty} \frac{1}{\pi} \left( \frac{\left( \frac{a}{b} \right)^{2m+1}}{(1 + \left( \frac{a}{b} \right)^{2m+1})^2} \right).
\]

Since the function \( \frac{n}{1 + n^2 \pi^2} \) is an increasing function on \((0, 1)\), the jump rate \( J_{i,j}(\theta, \varphi) \) of \( \theta - \varphi = \frac{1}{2} \pi + n \pi \) is bigger than that of \( \theta - \varphi = (2n + 1) \pi \). For the jump rate \( J_{i,i}(\theta, \varphi) \), we can see the value of jump rate dependence of the ratio of \( a \) to \( b \).

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References


Jump measure densities corresponding to Brownian motion on an annulus

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Abstract

We consider the jump measure densities for Dirichlet forms of a non-local type corresponding to the skew product diffusion processes of a one dimensional diffusion process on $\mathbb{R}$ and the spherical Brownian motion on $S^{d-1}$. In [7], we showed a limit theorem for the Dirichlet forms of local type to that of non-local type, in view of semi groups for time changes of these skew product. Further, the Dirichlet forms corresponding to the limit processes are obtained in [8]. The Dirichlet form corresponding to the limit process has a diffusion part, a jump part, and a killing part. In this paper we discuss the jump rate corresponding to the time changed skew product diffusion process of an extended Bessel process and the spherical Brownian motion. We focus on a 2 dimensional case so that the corresponding skew product diffusion processes is recurrent. We can find the effects of the recurrent property to jump rates. We clarify jump measure densities corresponding to Brownian motion on annulus.